8.5 Finding the Area Between Curves Expressed as Functions of y

Name: _____ Class: _____ Date: _____ **Total: 20 marks**

KEY WORDS horizontal strip • with respect to y y • right curve • left curve • region • limits

Objective

Build the skills to answer exam questions on **the area between curves as functions of y** y : right minus left, limits on the y-axis, and choosing the easier variable.

You must be able to:

- write curves in the form $x = g(y)$
- find the y-values where curves meet
- set up $\int_c^d (\text{right} - \text{left})\, dy$
- decide when integrating in y is simpler than in x

1 Key knowledge and worked examples

■ Horizontal strips

step	what to do
1	write each boundary as $x = g(y)$
2	solve for the y-values where the curves meet: these are the limits
3	$A = \int_c^d \left(x_{\text{right}} - x_{\text{left}}\right) dy$
when to use y	the left and right boundaries stay the same, but top and bottom would change
curves swap sides	split at the crossing, as with x

Worked example 1: line and parabola. $x = y^2$ and $x = y + 2$.

1. Known: $y^2 = y + 2$ at $y = -1$ and $y = 2$; the line is on the right.

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2. Rule: $\displaystyle\int_{-1}^{2} (y + 2 - y^2)\, dy$.
3. Numbers: $\left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$.
4. Result: $\frac{9}{2}$.

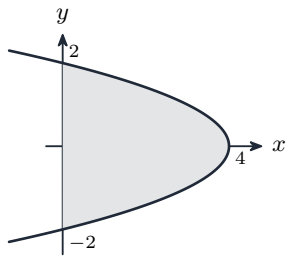
Worked example 2: switch variable. The region between $y = \sqrt{x}$, the y-axis and $y = 2$.

1. Known: $x = y^2$ on the right, $x = 0$ on the left, $0 \le y \le 2$.
2. Rule: $\displaystyle\int_{0}^{2} y^2\, dy$.
3. Numbers: $\frac{8}{3}$.
4. Result: $\frac{8}{3}$.

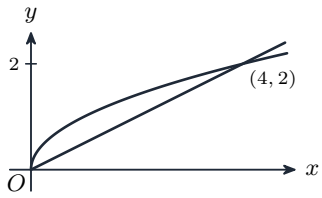
Answer shapes: label the limits as y-values, and write dy.

2 Practice

2.1 Find the area of the region between the curve $x = 4 - y^2$ and the y-axis. [2]



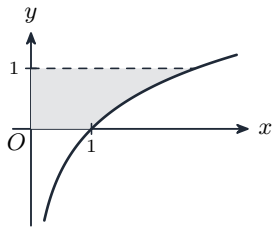
2.2 Find the area of the region between $x = y^2$ and $x = 2y$. [3]



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3 Exam-style questions

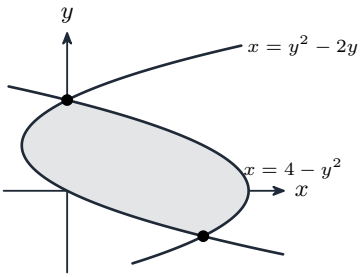
3.1 Find the area of the region bounded by $y = \ln x$, the y-axis, $y = 0$ and $y = 1$. [3]



3.2 R is the region enclosed by the curves $x = 4 - y^2$ and $x = y^2 - 2y$.

(a) Find the coordinates of the points where the curves meet. [2]

(b) Find the area of R. [3]



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(c) The horizontal line $y = k$ divides R into two parts of equal area. Find k. [3]

(d) Find the area of the region between the two curves for $2 \leq y \leq 3$. [4]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

- $\int_{-2}^{2} (4 - y^2)\, dy$
- $= \mathbf{\frac{32}{3}}$

2.2

- they meet at $y = 0$ and $y = 2$; $2y$ is on the right
- $\int_{0}^{2} (2y - y^2)\, dy = 4 - \frac{8}{3}$
- $= \mathbf{\frac{4}{3}}$

3.1

- $x = e^y$
- $\int_{0}^{1} e^y\, dy$
- $= \mathbf{e - 1}$

3.2

(a) $y^2 - y - 2 = 0$, $y = -1$ or $y = 2$; $(\mathbf{3}, \mathbf{-1})$ and $(\mathbf{0}, \mathbf{2})$

(b) $\int_{-1}^{2} (4 + 2y - 2y^2)\, dy = \left[4y + y^2 - \frac{2}{3}y^3\right]_{-1}^{2} = \frac{20}{3} + \frac{7}{3} = \mathbf{9}$

(c) $\int_{-1}^{k} (4 + 2y - 2y^2)\, dy = \frac{9}{2}$; $4k^3 - 6k^2 - 24k + 13 = 0$; $\mathbf{k = \frac{1}{2}}$ (the width $4 + 2y - 2y^2$ is symmetric about $y = \frac{1}{2}$)

(d) for $y > 2$, $y^2 - 2y$ is on the right; $\int_{2}^{3} (2y^2 - 2y - 4)\, dy$; $\left[\frac{2}{3}y^3 - y^2 - 4y\right]_{2}^{3} = -3 + \frac{20}{3}$ $= \mathbf{\frac{11}{3}}$

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