8.4 Finding the Area Between Curves Expressed as Functions of x

Name: _____ Class: _____ Date: _____ **Total: 20 marks**

KEY WORDS area between curves • bottom curve • region • symmetric • intersection • top curve

Objective

Build the skills to answer exam questions on **the area between curves as functions of x** x : top minus bottom, finding intersections, splitting where curves cross, and dividing a region.

You must be able to:

- find where two curves meet
- set up $\int_a^b (\text{top} - \text{bottom})\,dx$
- split the integral where the top and bottom curves swap
- find a line that divides a region into equal areas

1 Key knowledge and worked examples

■ Area between curves

step	what to do
1 sketch	see which curve is on **top**
2 limits	solve $f(x) = g(x)$ for the x-values where they meet
3 integral	$A = \int_a^b \big(\text{top} - \text{bottom}\big)\,dx$
curves cross inside	split at each crossing, so every piece is positive
check	an area is always positive

Worked example 1: line and parabola. $y = x + 2$ and $y = x^2$.

1. Known: $x^2 = x + 2$ at $x = -1$ and $x = 2$; the line is on top.
2. Rule: $\displaystyle\int_{-1}^{2} (x + 2 - x^2)\,dx$.
3. Numbers: $\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{10}{3} + \frac{7}{6}$.
4. Result: $\frac{9}{2}$.

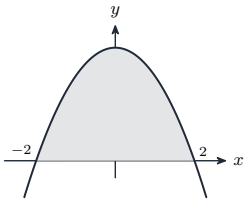
Worked example 2: trig. Between $y = \cos x$ and $y = \sin x$ for $0 \le x \le \frac{\pi}{4}$.

1. Known: $\cos x \ge \sin x$ there.
2. Rule: $\displaystyle\int_0^{\pi/4} (\cos x - \sin x)\,dx$.
3. Numbers: $\big[\sin x + \cos x\big]_0^{\pi/4} = \sqrt{2} - 1$.
4. Result: $\sqrt{2} - 1$.

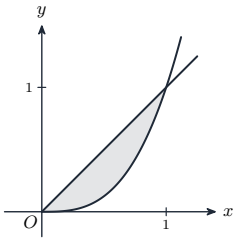
Answer shapes: write the integral with its limits before you evaluate it.

2 Practice

2.1 Find the area of the region between $y = 4 - x^2$ and the x-axis. [2]



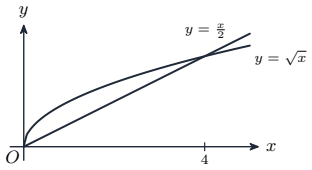
2.2 Find the area of the region between $y = x$ and $y = x^3$ for $0 \le x \le 1$. [3]



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3 Exam-style questions

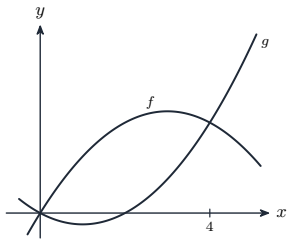
3.1 Find the area of the region bounded by $y = \sqrt{x}$ and $y = \dfrac{x}{2}$. [3]



3.2 Let $f(x) = 6x - x^2$ and $g(x) = x^2 - 2x$. The graphs are shown.

(a) Find the x-coordinates of the points where the graphs meet. [2]

(b) Find the area of the region enclosed by the two graphs. [3]



(c) The vertical line $x = k$ divides this region into two parts of equal area. Find k. [3]

(d) Find the total area between the two graphs for $0 \leq x \leq 5$. [4]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

- $\int_{-2}^{2} (4 - x^2)\,dx$
- $= \mathbf{\frac{32}{3}}$

2.2

- $x \geq x^3$ on $[0, 1]$: $\int_{0}^{1} (x - x^3)\,dx$
- $\frac{1}{2} - \frac{1}{4}$
- $= \mathbf{\frac{1}{4}}$

3.1

- they meet at $x = 0$ and $x = 4$
- $\int_{0}^{4} \left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{16}{3} - 4$
- $= \mathbf{\frac{4}{3}}$

3.2

(a) $6x - x^2 = x^2 - 2x$, $2x(x - 4) = 0$; $\mathbf{x = 0}$ and $\mathbf{x = 4}$

(b) $\int_{0}^{4} (8x - 2x^2)\,dx = \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = \mathbf{\frac{64}{3}}$

(c) $\int_{0}^{k} (8x - 2x^2)\,dx = \frac{32}{3}$; $k^3 - 6k^2 + 16 = 0$; $\mathbf{k = 2}$ (the region is symmetric about $x = 2$)

(d) g is on top for $4 < x < 5$, so split at 4; $\int_{4}^{5} (2x^2 - 8x)\,dx = \frac{14}{3}$; $\frac{64}{3} + \frac{14}{3} = \mathbf{26}$

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