7.8 Exponential Models with Differential Equations

Name: _____ Class: _____ Date: _____ **Total: 26 marks**

KEY WORDS exponential growth • exponential decay • half-life • doubling time • Newton's law of cooling • dose

Objective

Build the skills to answer exam questions on **exponential models with differential equations** : $\frac{dy}{dt} = ky$, doubling time and half-life, Newton's law of cooling, and repeated doses.

You must be able to:

- solve $\frac{dy}{dt} = ky$ as $y = y_0 e^{kt}$
- find k from a doubling time, a half-life or two data values
- use $T = A + (T_0 - A)e^{kt}$ for cooling and heating
- answer "when" questions with logarithms

1 Key knowledge and worked examples

■ Exponential models

model	solution
$\frac{dy}{dt} = ky$	$y = y_0 e^{kt}$; growth if $k > 0$, decay if $k < 0$
doubling time D	$k = \frac{\ln 2}{D}$
half-life H	$k = -\frac{\ln 2}{H}$, so $y = y_0 \left(\frac{1}{2}\right)^{t/H}$
two values	$y(t_1) = y_0 e^{kt_1}$ gives $k = \frac{1}{t_1} \ln \frac{y(t_1)}{y_0}$
cooling, room temperature A	$\frac{dT}{dt} = k(T - A)$ gives $T = A + (T_0 - A)e^{kt}$
"when?"	isolate e^{kt}, then take ln

Worked example 1: doubling. Bacteria double every 3 hours, starting from 500. Find the number after 7 hours.

1. Known: $k = \frac{\ln 2}{3}$.
2. Rule: $y = 500e^{kt} = 500 \cdot 2^{t/3}$.
3. Numbers: $500 \cdot 2^{7/3}$.
4. Result: **about** 2520 **bacteria**.

Worked example 2: half-life. Carbon-14 has half-life 5730 years. What fraction is left after 2000 years?

1. Known: $y = y_0 \left(\frac{1}{2}\right)^{t/5730}$.
2. Rule: fraction $= \frac{y}{y_0}$.
3. Numbers: $\left(\frac{1}{2}\right)^{2000/5730}$.
4. Result: **about** 0.785.

Answer shapes: give k exactly (with ln) and then as a decimal; give units for times.

2 Practice

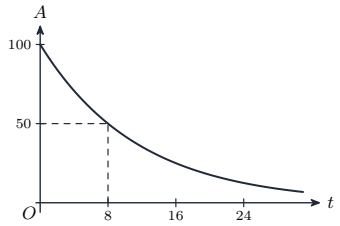
2.1 $\frac{dy}{dt} = 0.04y$ and $y(0) = 200$. Find $y(t)$ and $y(10)$. [2]

$\frac{dy}{dt} = 0.04y$ $y(0) = 200$

2.2 A population grows according to $\frac{dP}{dt} = kP$. It grows from 1000 to 1500 in 4 years. Find k, and the population after 10 years. [3]

$P(0) = 1000$ $P(4) = 1500$

2.3 Iodine-131 has a half-life of 8 days. A sample starts with 100 mg; its graph is shown. After how many days does 10 mg remain? [3]



exercises.lol exercises.com.cn

3 Exam-style questions

3.1 A patient takes 400 mg of a medicine at $t = 0$. The amount A mg in the body satisfies $\dfrac{dA}{dt} = -kA$, and the half-life of the medicine is 6 hours.

(a) Find k and an expression for $A(t)$. [3]

dose 400 mg | half-life 6 h

(b) Find the amount after 15 hours, and the time when 50 mg remain. [4]

(c) A second 400 mg dose is taken at $t = 6$. Find the amount just after the second dose, and the amount at $t = 12$ (before any third dose). Find the rate at which the amount is decreasing at $t = 12$. [5]

(d) The patient takes 400 mg every 6 hours. Let D_n be the amount just after the nth dose, so $D_1 = 400$. Show that $D_{n+1} = \frac{1}{2}D_n + 400$. Find D_2 and D_3, and the level D that the amounts approach, where $D = \frac{1}{2}D + 400$. [6]

exercises.lol exercises.com.cn

Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

- $\mathbf{y = 200e^{0.04t}}$
- $y(10) = 200e^{0.4} \approx \mathbf{298.4}$

2.2

- $1500 = 1000e^{4k}$
- $k = \frac{1}{4}\ln 1.5 \approx \mathbf{0.1014}$
- $P(10) = 1000(1.5)^{2.5} \approx \mathbf{2756}$

2.3

- $k = -\frac{\ln 2}{8}$
- $10 = 100e^{kt}$, so $t = \dfrac{\ln 0.1}{k}$
- $t \approx \mathbf{26.6}$ days

3.1

(a) $\frac{1}{2} = e^{-6k}$; $k = \frac{\ln 2}{6} \approx \mathbf{0.1155}$; $\mathbf{A = 400e^{-kt}} = 400\left(\frac{1}{2}\right)^{t/6}$

(b) $400\left(\frac{1}{2}\right)^{2.5} \approx \mathbf{70.7}$ mg; $50 = 400\left(\frac{1}{2}\right)^{t/6}$, $\left(\frac{1}{2}\right)^{t/6} = \frac{1}{8}$; $\mathbf{t = 18}$ hours

(c) at $t = 6$: 200 remain; $200 + 400 = \mathbf{600}$ mg; at $t = 12$: **300** mg; rate $= kA = \frac{\ln 2}{6}(300)$ $\approx \mathbf{34.7}$ mg per hour

(d) in 6 hours the amount halves; so just before dose $n + 1$ there is $\frac{1}{2}D_n$, then 400 is added; $D_2 = 600$, $D_3 = 700$; $D = \frac{1}{2}D + 400$; $\mathbf{D = 800}$ mg

exercises.lol exercises.com.cn