

# 6.13 Evaluating Improper Integrals

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 13 marks

**KEY WORDS** diverges • improper integral • limit • converges  
• integral

## Objective

Build the skills to answer exam questions on **improper integrals** – integrals with an infinite limit or an infinite discontinuity.

**You must be able to:**

- rewrite an improper integral as a **limit**
- evaluate the limit to decide **convergence** or **divergence**
- handle a discontinuity inside the interval

## 1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later – follow the steps and you can do the Practice and Exam-style questions yourself.

### ■ Infinite upper limit

Replace  $\infty$  with  $b$  and take a limit:

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx = \lim_{b \rightarrow \infty} \left[ -\frac{1}{x} \right]_1^b = \lim_{b \rightarrow \infty} \left( -\frac{1}{b} + 1 \right) = 1.$$

The limit is finite, so the integral **converges** to 1.

### ■ A divergent example

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} [\ln x]_1^b = \lim_{b \rightarrow \infty} \ln b = \infty. \text{ The limit is infinite, so it } \mathbf{diverges}.$$

### ■ Discontinuity at an endpoint

$$\int_0^1 \frac{1}{\sqrt{x}} dx \text{ is improper at } x = 0: \lim_{a \rightarrow 0^+} \int_a^1 x^{-1/2} dx = \lim_{a \rightarrow 0^+} [2\sqrt{x}]_a^1 = 2 - 0 = 2.$$

Converges.

■ The rule for  $1/x^p$  on  $[1, \infty)$

$\int_1^\infty \frac{1}{x^p} dx$  **converges** when  $p > 1$  and **diverges** when  $p \leq 1$  – a handy check.

## 2 Practice

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Now apply the methods above.

**2.1** Rewrite  $\int_1^\infty \frac{1}{x^3} dx$  as a limit. [1]

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**2.2** Evaluate  $\int_1^\infty \frac{1}{x^3} dx$ . [3]

**2.3** State whether  $\int_1^\infty \frac{1}{x^{1/2}} dx$  converges or diverges (use the  $p$ -rule). [1]

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## 3 Exam-style questions

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**3.1**  $\int_1^\infty \frac{1}{x^p} dx$  converges when [1]

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**A**  $p < 1$

**B**  $p = 1$

**C**  $p > 1$

**D** all  $p$

**3.2** Evaluate  $\int_0^{\infty} e^{-x} dx$ .

(a) Write it as a limit. [1]

(b) Evaluate. [3]

**3.3** Determine whether  $\int_2^{\infty} \frac{1}{x} dx$  converges or diverges, showing your reasoning with a limit. [3]

## Solutions

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Each answer shows the steps, then the **result**.

### 2.1

- $\lim_{b \rightarrow \infty} \int_1^b x^{-3} dx$

### 2.2

- $\left[-\frac{1}{2x^2}\right]_1^b = -\frac{1}{2b^2} + \frac{1}{2}$ ; as  $b \rightarrow \infty$  this  $\rightarrow \frac{1}{2}$

### 2.3

- Diverges –  $p = \frac{1}{2} \leq 1$

### 3.1 C

- converges for  $p > 1$

### 3.2

(a)

- $\lim_{b \rightarrow \infty} \int_0^b e^{-x} dx$

(b)

- $\left[-e^{-x}\right]_0^b = -e^{-b} + 1$ ;  $\rightarrow 1$  as  $b \rightarrow \infty$

### 3.3

- $\lim_{b \rightarrow \infty} [\ln x]_2^b = \lim_{b \rightarrow \infty} (\ln b - \ln 2)$ ;  $\ln b \rightarrow \infty$ , so it **diverges**