

10.5 Harmonic Series and p-Series

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS harmonic series • converges • diverges

Objective

Build the skills to answer exam questions on the **harmonic series** and **p-series**.

You must be able to:

- apply the **p-series** rule: $\sum \frac{1}{n^p}$ converges iff $p > 1$
- recognise the **harmonic series** ($p = 1$) as divergent
- use p-series as the yardstick for comparison tests

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later – follow the steps and you can do the Practice and Exam-style questions yourself.

■ The p-series rule

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1, \text{ diverges if } p \leq 1.$$

■ The harmonic series

The special case $p = 1$, $\sum \frac{1}{n}$, is the **harmonic series** – it **diverges** even though its terms shrink to 0. A must-know fact.

■ Applying the rule

- $\sum \frac{1}{n^2}$: $p = 2 > 1$, **converges**.
- $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$: $p = \frac{1}{2} \leq 1$, **diverges**.
- $\sum \frac{1}{n^{3/2}}$: $p = \frac{3}{2} > 1$, **converges**.

■ The yardstick

p-series are the standard family you compare unknown series to (next subtopic). Know the rule instantly.

2 Practice

Now apply the methods above.

2.1 State whether $\sum \frac{1}{n^4}$ converges. [1]

2.2 State whether $\sum \frac{1}{n^{2/3}}$ converges. [1]

2.3 Does the harmonic series converge? Give the value of p . [2]

3 Exam-style questions

3.1 $\sum \frac{1}{n^p}$ converges if and only if [1]

- ☐ **A** $p < 1$
B $p = 1$
C $p > 1$
D $p \geq 0$

3.2 Classify each as convergent or divergent, stating p :

(a) $\sum \frac{1}{n^{1.2}}$ [1]

(b) $\sum \frac{1}{n^{0.9}}$ [1]

(c) $\sum \frac{1}{n}$ [1]

3.3 Explain why $\sum \frac{1}{\sqrt{n}}$ diverges but $\sum \frac{1}{n^2}$ converges, in terms of p . [2]

Solutions

Each answer shows the steps, then the **result**.

2.1

- Converges ($p = 4 > 1$)

2.2

- Diverges ($p = \frac{2}{3} \leq 1$)

2.3

- No, it diverges; $p = 1$

3.1 C

- converges iff $p > 1$

3.2

(a)

- convergent, $p = 1.2 > 1$

(b)

- divergent, $p = 0.9 \leq 1$

(c)

- divergent, $p = 1$

3.3

- $\sum_{p=1} \frac{1}{\sqrt{p}}$ has $p = \frac{1}{2} \leq 1$ (diverges); $\sum \frac{1}{n^2}$ has $p = 2 > 1$ (converges) – the boundary is