10.3 The nth Term Test for Divergence

Name: _____ Class: _____ Date: _____ **Total: 12 marks**

KEY WORDS harmonic series • diverges • converges

Objective

Build the skills to answer exam questions on the **nth term test for divergence**.

You must be able to:

- apply: if $\lim_{n\to\infty} a_n \neq 0$, the series **diverges**
- recognise that the test is only a test for **divergence**
- know that $a_n \to 0$ does **not** prove convergence

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later – follow the steps and you can do the Practice and Exam-style questions yourself.

■ The test

If the terms do not shrink to zero, the sum cannot settle:

$$\lim_{n\to\infty} a_n \neq 0 \;\Rightarrow\; \sum a_n \text{ diverges.}$$

■ A divergent example

$\sum \frac{n}{n+1}$: $\frac{n}{n+1} \to 1 \neq 0$, so the series **diverges** by the nth term test.

■ It only shows divergence

If $\lim a_n = 0$, the test is **inconclusive** – the series may converge or diverge. The harmonic series $\sum \frac{1}{n}$ has terms $\to 0$ yet **diverges**.

■ First check to do

Because it is quick, always apply the nth term test **first**. Only if the limit is 0 do you move on to another test.

2 Practice

Now apply the methods above.

2.1 Find $\lim_{n\to\infty} \frac{2n}{3n+1}$. [2]

2.2 State whether $\sum \frac{2n}{3n+1}$ converges or diverges, with a reason. [2]

2.3 For $\sum \frac{1}{n}$, the terms go to 0. Does the nth term test prove convergence? [1]

3 Exam-style questions

3.1 If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ [1]

- ☐ **A** converges
- **B** diverges
- **C** may converge or diverge (test inconclusive)
- **D** equals 0

3.2 Consider $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+5}$.

(a) Find $\lim_{n\to\infty} a_n$. [2]

(b) State the conclusion of the nth term test. [2]

3.3 A student claims that because the terms of $\sum \frac{1}{\sqrt{n}}$ tend to 0, the series converges. Explain the error. [2]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\dfrac{2n}{3n+1} \to \dfrac{2}{3}$

2.2

- Diverges – the limit is $\frac{2}{3} \neq 0$, so by the nth term test it diverges

2.3

- No – the test is inconclusive when the terms go to 0

3.1 C

- the test is inconclusive when $a_n \to 0$

3.2

(a)

- $\dfrac{n^2}{2n^2+5} \to \dfrac{1}{2}$

(b)

- Limit $\neq 0$, so the series **diverges**

3.3

- The nth term test only detects **divergence**; $a_n \to 0$ does not prove convergence (in fact $\sum \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \leq 1$ and diverges)