AP CALCULUS BC • EXERCISE SHEET

1.15 Connecting Limits at Infinity and Horizontal Asymptotes

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS horizontal asymptote • degrees • ratio of the leading coefficients • limit at infinity • limit

Objective

Build the skills to answer exam questions on **connecting limits at infinity and horizontal asymptotes**.

You must be able to:

- relate a **limit at infinity** to a **horizontal asymptote**
- find it from the ratio of leading terms

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later.

■ Limits at infinity

$\lim_{x\to\infty} f(x) = L$ means $y = L$ is a horizontal asymptote. For a rational function, compare leading terms:

- degree(top) < degree(bottom) $\to y = 0$;
- degrees equal $\to y = \dfrac{\text{leading coeff of top}}{\text{leading coeff of bottom}}$;
- degree(top) > degree(bottom) $\to$ no horizontal asymptote.

■ Example

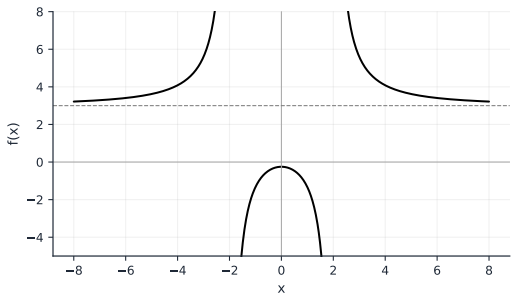
$\lim_{x\to\infty} \dfrac{1}{x} = 0$, so $y = 0$ is a horizontal asymptote.

■ Key knowledge: Read end behaviour from leading powers

For a rational limit at infinity, divide numerator and denominator by the highest denominator power. Terms containing negative powers then vanish. A horizontal asymptote concerns end behaviour, not every finite input.

■ Worked example: Read end behaviour from leading powers

For $f(x) = (3x^2+1)/(x^2-4)$, division by x^2 gives $(3+1/x^2)/(1-4/x^2)$. As $x \to \pm\infty$, its limit is 3. Thus $y = 3$ is a horizontal asymptote at both ends, even though the graph has vertical asymptotes at finite inputs.



2 Practice

2.1 State what $\lim_{x\to\infty} f(x) = L$ indicates. [1]

2.2 For $f(x) = \dfrac{1}{x}$, state $\lim_{x\to\infty} f(x)$. [1]

2.3 For $f(x) = \dfrac{2x}{x+1}$, state the horizontal asymptote. [2]

3 Exam-style questions

3.1 $\lim_{x\to\infty} f(x) = L$ means $y = L$ is a [1]

- **A** vertical asymptote
- **B** horizontal asymptote
- **C** hole
- **D** root

3.2 $\lim_{x\to\infty} \dfrac{1}{x}$ equals [1]

- **A** 1
- **B** 0
- **C** ∞
- **D** -1

3.3 $f(x) = \dfrac{3x^2}{x^2+1}$.

(a) Compare the degrees. [1]

(b) Take the ratio of the leading coefficients. [1]

(c) State the horizontal asymptote. [1]

3.4 (a) Find $\lim_{x\to\infty}(2x^2 - x)/(5x^2 + 1)$, showing your algebraic method. [2]

(b) State the horizontal asymptote and explain whether it determines the value at $x = 0$. [2]

Solutions

Each answer shows the steps, then the **result**.

2.1

- the graph has a horizontal asymptote $y = L$

2.2

- 0

2.3

- $y = 2$ (equal degrees, ratio 2/1)

3.1 B

- horizontal asymptote.

3.2 B

- 0.

3.3

(a)

- equal (both degree 2)

(b)

- $3/1 = 3$

(c)

- $y = 3$

3.4

(a) Divide by x^2: the limit is $(2 - 1/x)/(5 + 1/x^2) \to 2/5$.

(b) $y = 2/5$; it describes end behaviour, whereas direct substitution gives $g(0) = 0$.