

Exercise walk-through

9.9

Finding the Area of the Region Bounded by Two Polar Curves

AP Calculus BC

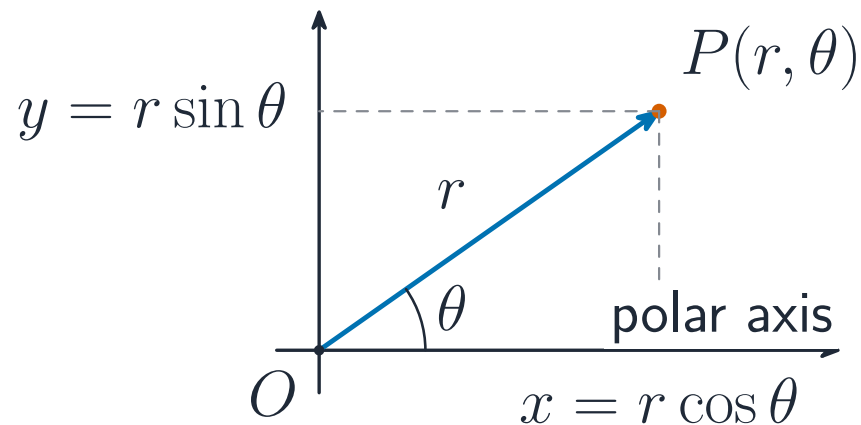
Polar coordinates

Polar coordinates locate a point by its distance r and angle θ :

$$x = r \cos \theta, \quad y = r \sin \theta.$$

A polar curve $r = f(\theta)$ is a **parametric curve in θ** , so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

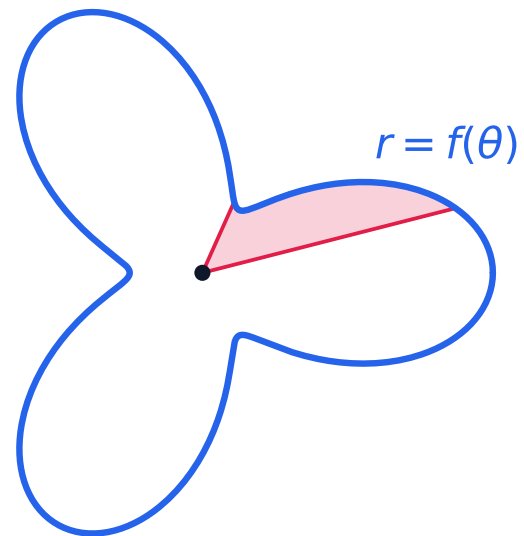


Area of a polar region

The region is a **fan** of thin triangular sectors, each of area $\frac{1}{2}r^2 d\theta$:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

Choosing the right θ -limits – where the curve **starts and finishes tracing** the region – is the main challenge.



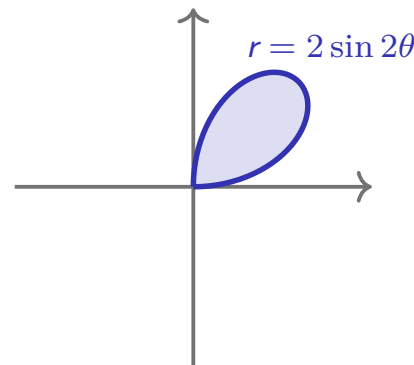
$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

Worked example – one petal of a rose

Area of one petal of $r = 2 \sin(2\theta)$, traced for $0 \leq \theta \leq \frac{\pi}{2}$

Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$:

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/2} (2 \sin 2\theta)^2 d\theta = \int_0^{\pi/2} (1 - \cos 4\theta) d\theta \\ &= \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{2}. \end{aligned}$$

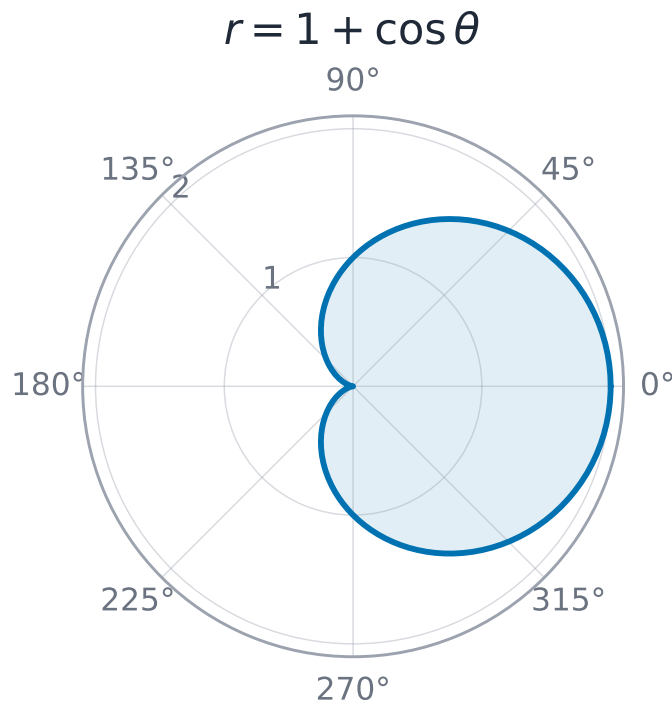


Between two polar curves

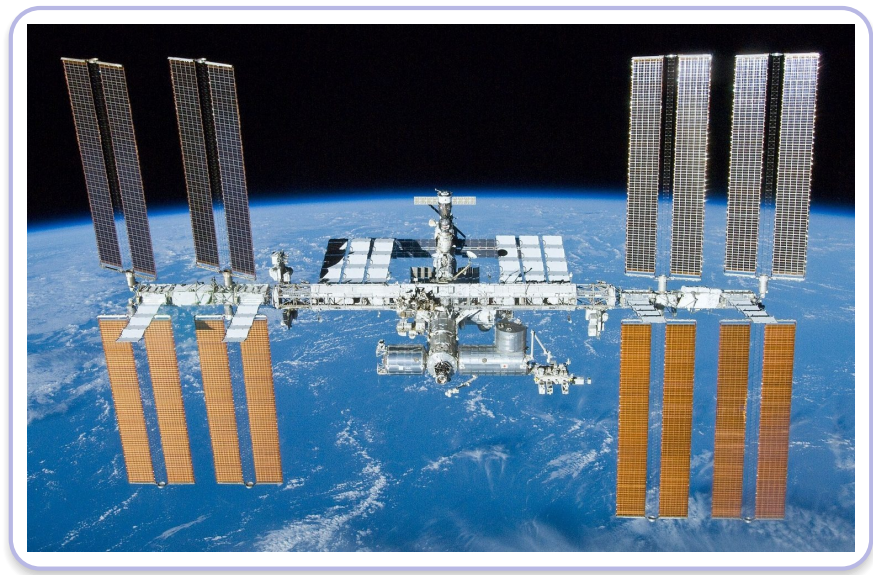
For the region between an outer $r_1(\theta)$ and an inner $r_2(\theta)$, subtract the fans:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta.$$

Subtract the **squares**, not the radii – and find where the curves **intersect** to set α and β .



Defining and Differentiating Parametric Equations



The ISS in orbit: parametric equations describe position as a function of time

Exercises 2.1-2.3

2.1 State the area-between-two-polar-curves formula.

[1]

2.2 Find the intersection angles of $r = 2$ and $r = 4 \cos \theta$.

[3]

2.3 Between $\theta = 0$ and $\theta = \frac{\pi}{6}$, which is larger: $r = 2$ or $r = 4 \cos \theta$?

[1]

Exercises 2.1-2.3

2.1 State the area-between-two-polar-curves formula.

[1]

2.1

- $$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta$$

2.2 Find the intersection angles of $r = 2$ and $r = 4 \cos \theta$.

[3]

2.2

- $$2 = 4 \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$$

2.3 Between $\theta = 0$ and $\theta = \frac{\pi}{6}$, which is larger: $r = 2$ or $r = 4 \cos \theta$?

[1]

2.3

- $r = 4 \cos \theta$ is larger (at $\theta = 0$: $4 > 2$)

Exercise 3.1

3.1 The area between an outer curve r_1 and inner r_2 is

[1]



A $\frac{1}{2} \int (r_1 - r_2)^2 d\theta$

B $\frac{1}{2} \int (r_1^2 - r_2^2) d\theta$

C $\int (r_1^2 - r_2^2) d\theta$

D $\frac{1}{2} \int (r_1^2 + r_2^2) d\theta$

Exercise 3.1

3.1 The area between an outer curve r_1 and inner r_2 is

[1]

- ☐ **A** $\frac{1}{2} \int (r_1 - r_2)^2 d\theta$
- B** $\frac{1}{2} \int (r_1^2 - r_2^2) d\theta$
- C** $\int (r_1^2 - r_2^2) d\theta$
- D** $\frac{1}{2} \int (r_1^2 + r_2^2) d\theta$

3.1 B

- half the integral of the difference of the squared radii

Exercise 3.2 (a)

3.2 Two curves are $r = 1$ (inner) and $r = 2 \cos \theta$ (outer) where they overlap.

(a) Find the intersection angles.

[2]

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3.2 Two curves are $r = 1$ (inner) and $r = 2 \cos \theta$ (outer) where they overlap.

(a) Find the intersection angles.

[2]

(a)

- $1 = 2 \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$

Exercise 3.2 (b)

3.2 Two curves are $r = 1$ (inner) and $r = 2 \cos \theta$ (outer) where they overlap.

(b) Set up the area inside $r = 2 \cos \theta$ and outside $r = 1$.

[3]

Exercise 3.2 (b)

3.2 Two curves are $r = 1$ (inner) and $r = 2 \cos \theta$ (outer) where they overlap.

(b) Set up the area inside $r = 2 \cos \theta$ and outside $r = 1$.

[3]

(b)

- $$A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} ((2 \cos \theta)^2 - 1^2) d\theta$$

Exercise 3.3

3.3 Explain why you must square the radii **before** subtracting, i.e. why $r_1^2 - r_2^2 \neq (r_1 - r_2)^2$ in the area formula. [2]

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3.3

- Each sector's area is $\frac{1}{2}r^2 d\theta$, so the region between is $\frac{1}{2}(r_1^2 - r_2^2) d\theta$ – a difference of the two **squared** radii; $(r_1 - r_2)^2$ would wrongly square the difference of the radii instead