

Exercise walk-through

9.8

Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve

AP Calculus BC

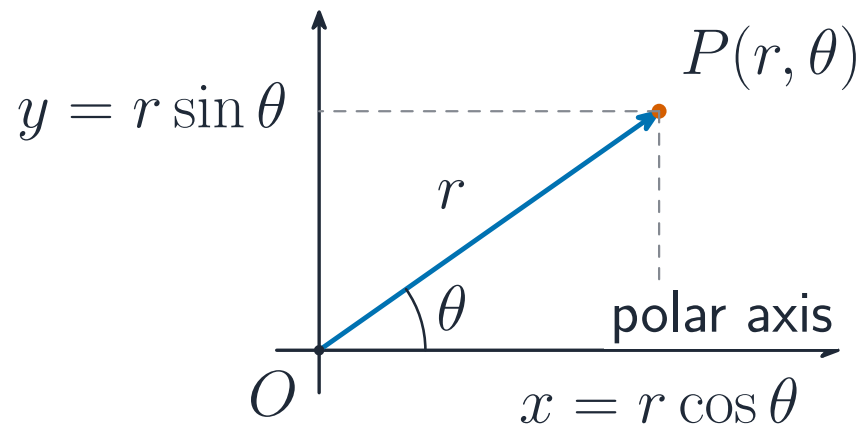
Polar coordinates

Polar coordinates locate a point by its distance r and angle θ :

$$x = r \cos \theta, \quad y = r \sin \theta.$$

A polar curve $r = f(\theta)$ is a **parametric curve in θ** , so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

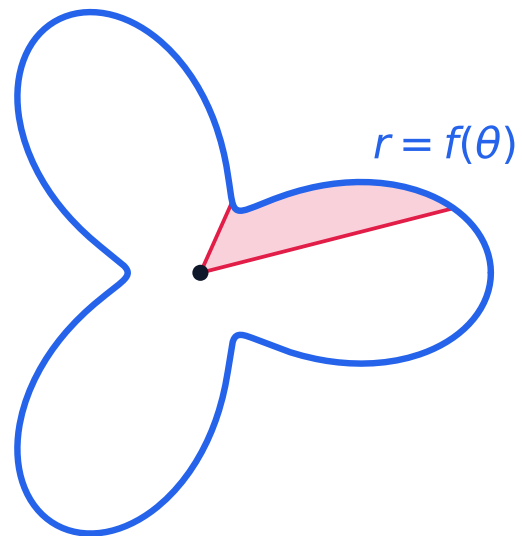


Area of a polar region

The region is a **fan** of thin triangular sectors, each of area $\frac{1}{2}r^2 d\theta$:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

Choosing the right θ -limits – where the curve **starts and finishes tracing** the region – is the main challenge.



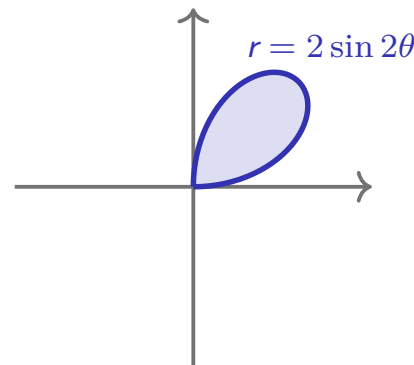
$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

Worked example – one petal of a rose

Area of one petal of $r = 2 \sin(2\theta)$, traced for $0 \leq \theta \leq \frac{\pi}{2}$

Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$:

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/2} (2 \sin 2\theta)^2 d\theta = \int_0^{\pi/2} (1 - \cos 4\theta) d\theta \\ &= \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{2}. \end{aligned}$$

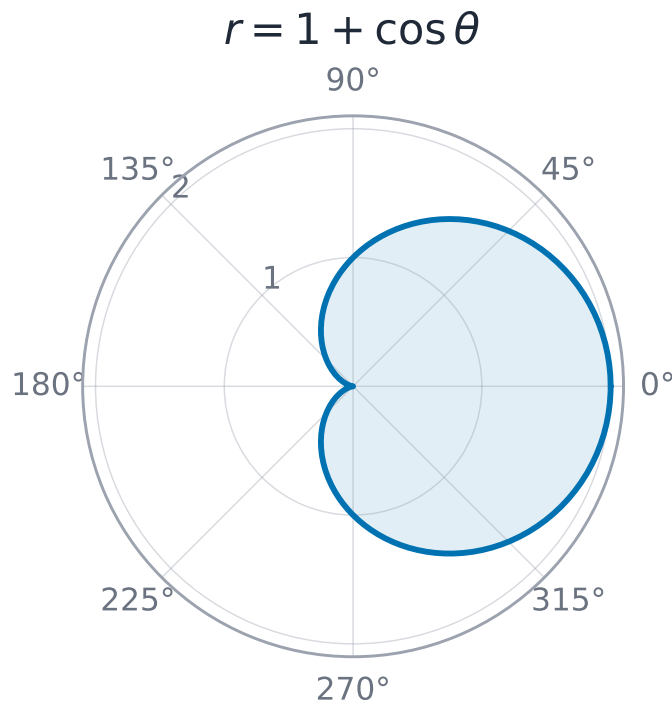


Between two polar curves

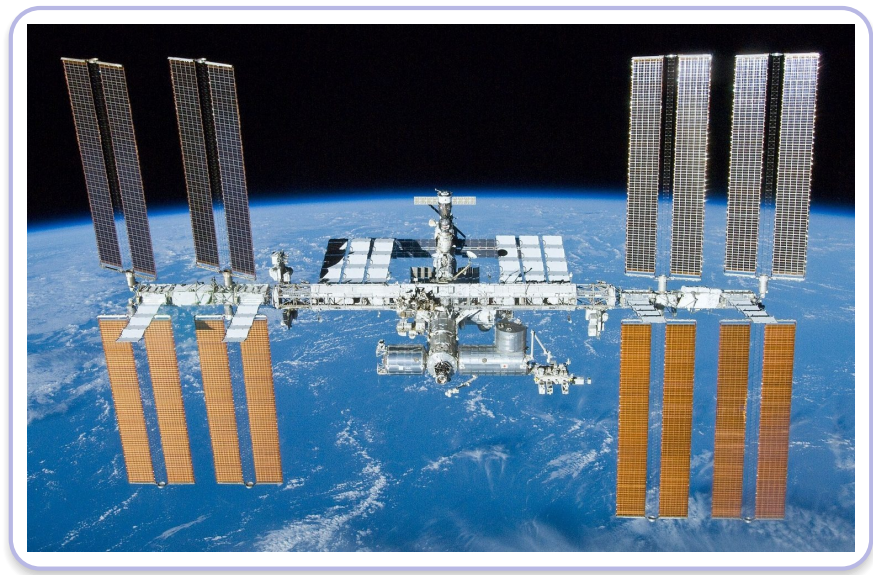
For the region between an outer $r_1(\theta)$ and an inner $r_2(\theta)$, subtract the fans:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta.$$

Subtract the **squares**, not the radii – and find where the curves **intersect** to set α and β .



Defining and Differentiating Parametric Equations



The ISS in orbit: parametric equations describe position as a function of time

Exercises 2.1-2.2

2.1 State the polar area formula.

[1]

2.2 Set up the area enclosed by $r = 3$ (a full circle).

[2]

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2.1 State the polar area formula.

[1]

2.1

- $A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta$

2.2 Set up the area enclosed by $r = 3$ (a full circle).

[2]

2.2

- $A = \frac{1}{2} \int_0^{2\pi} 3^2 d\theta$

Exercises 2.3·3.1

2.3 Evaluate $\frac{1}{2} \int_0^{2\pi} 9 \, d\theta$.

[2]

3.1 The area of a polar region is

[1]



A $\int_{\alpha}^{\beta} f(\theta) \, d\theta$

B $\frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta$

C $\pi \int_{\alpha}^{\beta} f(\theta) \, d\theta$

D $\frac{1}{2} \int_{\alpha}^{\beta} f(\theta) \, d\theta$

Exercises 2.3·3.1

2.3 Evaluate $\frac{1}{2} \int_0^{2\pi} 9 \, d\theta$.

[2]

2.3

- $\frac{1}{2}(9)(2\pi) = 9\pi$

3.1 The area of a polar region is

[1]

- ☐ **A** $\int_{\alpha}^{\beta} f(\theta) \, d\theta$
- B** $\frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta$
- C** $\pi \int_{\alpha}^{\beta} f(\theta) \, d\theta$
- D** $\frac{1}{2} \int_{\alpha}^{\beta} f(\theta) \, d\theta$

3.1 B

- half the integral of r^2

Exercise 3.2 (a)

3.2 Consider $r = 1 + \cos \theta$ (a cardioid), traced once for $0 \leq \theta \leq 2\pi$.

(a) Set up the area integral.

[2]

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(a) Set up the area integral.

[2]

(a)

- $A = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta$

Exercise 3.2 (b)

3.2 Consider $r = 1 + \cos \theta$ (a cardioid), traced once for $0 \leq \theta \leq 2\pi$.

(b) Expand $(1 + \cos \theta)^2$ ready to integrate.

[2]

Exercise 3.2 (b)

3.2 Consider $r = 1 + \cos \theta$ (a cardioid), traced once for $0 \leq \theta \leq 2\pi$.

(b) Expand $(1 + \cos \theta)^2$ ready to integrate.

[2]

(b)

- $(1 + \cos \theta)^2 = 1 + 2 \cos \theta + \cos^2 \theta$

Exercise 3.3

3.3 One petal of the rose $r = 2 \sin(3\theta)$ is traced as θ goes from 0 to $\frac{\pi}{3}$. Set up the area integral for one petal. [3]

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3.3

- $$A = \frac{1}{2} \int_0^{\pi/3} (2 \sin(3\theta))^2 d\theta = \frac{1}{2} \int_0^{\pi/3} 4 \sin^2(3\theta) d\theta$$