

Exercise walk-through

9.3

Finding Arc Lengths of Curves Given by Parametric Equations

AP Calculus BC

Differentiating a parametric curve

Parametric equations give x and y each as functions of a **parameter** t – so the curve **need not be a function of x** .

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (dx/dt \neq 0).$$

It is the chain rule: the dt cancels.

$x = t^2, y = t^3 - t$: **slope at $t = 2$**

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 1:$$

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t} \xrightarrow{t=2} \frac{11}{4}.$$

The second derivative – a classic trap

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

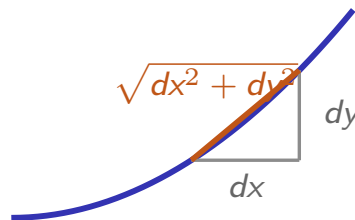
Differentiate the **first derivative** with respect to t , then divide by dx/dt **again**. Use it to test **concavity** of a parametric curve.

The ISS in orbit: parametric equations describe position as a function of time

Arc length of a parametric curve

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

This is Pythagoras: sum tiny hypotenuses $\sqrt{dx^2 + dy^2}$ over the parameter.



Exercises 2.1-2.3

2.1 State the parametric arc-length formula.

[1]

2.2 For $x = t^2$, $y = t^3$, write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$.

[2]

2.3 For $x = 3t$, $y = 4t$ on $[0, 2]$, set up and evaluate the arc length.

[3]

Exercises 2.1·2.3

2.1 State the parametric arc-length formula.

[1]

2.1

- $$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$$

2.2 For $x = t^2$, $y = t^3$, write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$.

[2]

2.2

- $$(2t)^2 + (3t^2)^2 = 4t^2 + 9t^4$$

2.3 For $x = 3t$, $y = 4t$ on $[0, 2]$, set up and evaluate the arc length.

[3]

2.3

- $$\sqrt{9 + 16} = 5; L = \int_0^2 5 dt = 10$$

Exercise 3.1

3.1 The parametric arc length uses the integrand

[1]

☐

A $\sqrt{1 + (dy/dx)^2}$

B $\sqrt{(dx/dt)^2 + (dy/dt)^2}$

C $\frac{dy/dt}{dx/dt}$

D $(dx/dt)(dy/dt)$

Exercise 3.1

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3.1 B

- the parametric integrand is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$

Exercise 3.2 (a)

3.2 A curve is $x = \cos t$, $y = \sin t$ for $0 \leq t \leq \pi$.

(a) Show the integrand simplifies to 1.

[2]

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(a) Show the integrand simplifies to 1.

[2]

(a)

- $(-\sin t)^2 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1$

Exercise 3.2 (b)

3.2 A curve is $x = \cos t$, $y = \sin t$ for $0 \leq t \leq \pi$.

(b) Evaluate the arc length.

[1]

Exercise 3.2 (b)

3.2 A curve is $x = \cos t$, $y = \sin t$ for $0 \leq t \leq \pi$.

(b) Evaluate the arc length.

[1]

(b)

- $\int_0^\pi 1 \, dt = \pi$

Exercise 3.3

3.3 Set up (do not evaluate) the arc-length integral for $x = t - \sin t$, $y = 1 - \cos t$ on $0 \leq t \leq 2\pi$. [3]

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3.3

- $\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t$; $L = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} dt$