

Exercise walk-through

# 8.9

## Volume with Disc Method: Revolving Around the x- or y-Axis

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AP Calculus BC

# Volumes with known cross sections

If a solid has a known **cross section** perpendicular to an axis, the volume is the integral of its **area**:

$$V = \int_a^b A(x) dx.$$

The exam phrase “the base of a solid... cross sections are squares” signals exactly this.

**square** of side  $s$ :  $A = s^2$

**rectangle**, height  $ks$ :  $A = ks^2$

equilateral **triangle**:  $A = \frac{\sqrt{3}}{4}s^2$

**semicircle**, diameter  $s$ :  $A = \frac{\pi}{8}s^2$

Set the shape's key length equal to the slice length  $s(x) = f(x) - g(x)$ , then integrate.

## Volumes of revolution – the disc method

Revolving a region about an axis makes a **solid of revolution**. If the region **touches** the axis, each slice is a **disc** :

$$V = \pi \int_a^b [r(x)]^2 dx.$$

About the  $y$ -axis: express  $x$  as a function of  $y$  and integrate in  $y$ .

# Disc, washer and shell

## Disc method

Rotating a region that touches the axis:  $V = \pi \int R(x)^2 dx$  – each slice is a solid circle.

## Washer method

The **washer method** is for a region with a gap:  $V = \pi \int (R^2 - r^2) dx$ , outer radius minus inner.

## Which variable

Slice perpendicular to the axis of rotation. Rotating about a horizontal axis integrates in  $x$ ; a vertical axis, in  $y$ .

## A shifted axis

Rotating about  $y = k$  changes the radii to  $|f(x) - k|$  – the method is unchanged.

Sketch the slice before writing the integral. Radius, thickness and limits all come from the picture.

## Exercises 2.1·2.3

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**2.1** Write the disc-method integral for revolving  $y = x$  on  $[0, 2]$  about the  $x$ -axis. [1]

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**2.2** Evaluate  $\pi \int_0^2 x^2 dx$ . [2]

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**2.3** Find the volume when  $y = \sqrt{x}$  on  $[0, 9]$  is revolved about the  $x$ -axis. [3]

## Exercises 2.1·2.3

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**2.1** Write the disc-method integral for revolving  $y = x$  on  $[0, 2]$  about the  $x$ -axis. [1]

**2.1**

- $\pi \int_0^2 x^2 dx$

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**2.2** Evaluate  $\pi \int_0^2 x^2 dx$ . [2]

**2.2**

- $\pi \left[ \frac{x^3}{3} \right]_0^2 = \frac{8\pi}{3}$

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**2.3** Find the volume when  $y = \sqrt{x}$  on  $[0, 9]$  is revolved about the  $x$ -axis. [3]

**2.3**

- $\pi \int_0^9 x dx = \pi \left[ \frac{x^2}{2} \right]_0^9 = \frac{81\pi}{2}$

## Exercise 3.1

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**3.1** Revolving  $y = f(x)$  about the  $x$ -axis gives the volume

[1]



**A**  $\int_a^b f(x) \, dx$

**B**  $\pi \int_a^b f(x) \, dx$

**C**  $\pi \int_a^b (f(x))^2 \, dx$

**D**  $2\pi \int_a^b f(x) \, dx$

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**3.1 C**

- the disc method integrates  $\pi(\text{radius})^2$

## Exercise 3.2 (a)

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**3.2** The region under  $y = x^2$  from  $x = 0$  to  $x = 2$  is revolved about the  $x$ -axis.

(a) Write the volume integral.

[2]

## Exercise 3.2 (a)

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**3.2** The region under  $y = x^2$  from  $x = 0$  to  $x = 2$  is revolved about the  $x$ -axis.

(a) Write the volume integral.

[2]

(a)

- $$\pi \int_0^2 (x^2)^2 dx = \pi \int_0^2 x^4 dx$$

## Exercise 3.2 (b)

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**3.2** The region under  $y = x^2$  from  $x = 0$  to  $x = 2$  is revolved about the  $x$ -axis.

(b) Evaluate it.

[3]

## Exercise 3.2 (b)

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**3.2** The region under  $y = x^2$  from  $x = 0$  to  $x = 2$  is revolved about the  $x$ -axis.

(b) Evaluate it.

[3]

(b)

- $\pi \left[ \frac{x^5}{5} \right]_0^2 = \frac{32\pi}{5}$

## Exercise 3.3

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**3.3** The region bounded by  $x = y^2$ ,  $y = 0$  to  $y = 2$ , and the  $y$ -axis is revolved about the  $y$ -axis. Find the volume. [4]

## Exercise 3.3

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**3.3** The region bounded by  $x = y^2$ ,  $y = 0$  to  $y = 2$ , and the  $y$ -axis is revolved about the  $y$ -axis. Find the volume. [4]

**3.3**

- radius =  $x = y^2$ ;  $V = \pi \int_0^2 (y^2)^2 dy = \pi \int_0^2 y^4 dy = \pi \left[ \frac{y^5}{5} \right]_0^2 = \frac{32\pi}{5}$