

Exercise walk-through

8.13

The Arc Length of a Smooth, Planar Curve and Distance Traveled

AP Calculus BC

The Arc Length of a Smooth Curve and Distance Traveled

The **arc length** of $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

This BC-only formula comes from su.

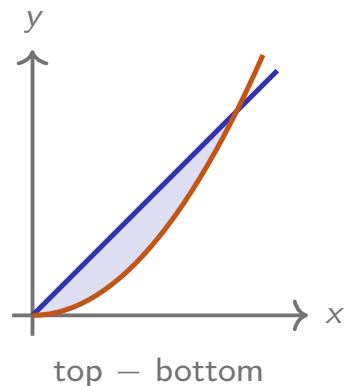
The same idea gives the distance a particle travels along a curved path.

Exam tips

- Area between curves is $\int (\text{top} - \text{bottom}) dx$ – find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a} \int_a^b f dx$.
- Accumulated change is $\int$ of a rate: total = initial value + $\int_a^b (\text{rate}) dt$.
- Integration means “adding up infinitely many tiny pieces” – set up the integrand as one thin slice.

Check yourself

- 1 Which integral gives total distance, $\int v$ or $\int |v|$?
- 2 Find the area between $y = x$ and $y = x^2$.
- 3 Write the average value of f on $[a, b]$.
- 4 A solid's cross sections are squares of side $s(x)$. Give V .



Answers 1. $\int |v|$ 2. $\frac{1}{6}$ 3. $\frac{1}{b-a} \int_a^b f$ 4. $V = \int_a^b [s(x)]^2 dx$

Exercises 2.1·2.3

2.1 State the arc-length formula for $y = f(x)$ on $[a, b]$. [1]

2.2 For $y = \frac{2}{3}x^{3/2}$, find $f'(x)$ and $(f'(x))^2$. [2]

2.3 Set up (do not evaluate) the arc length of $y = x^2$ from $x = 0$ to $x = 2$. [2]

Exercises 2.1-2.3

2.1 State the arc-length formula for $y = f(x)$ on $[a, b]$. [1]

2.1

- $$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

2.2 For $y = \frac{2}{3}x^{3/2}$, find $f'(x)$ and $(f'(x))^2$. [2]

2.2

- $$f'(x) = x^{1/2}; (f'(x))^2 = x$$

2.3 Set up (do not evaluate) the arc length of $y = x^2$ from $x = 0$ to $x = 2$. [2]

2.3

- $$f'(x) = 2x, \text{ so } L = \int_0^2 \sqrt{1 + 4x^2} dx$$

Exercise 3.1

3.1 The arc length of $y = f(x)$ on $[a, b]$ is

[1]



A $\int_a^b f(x) \, dx$

B $\int_a^b \sqrt{1 + (f'(x))^2} \, dx$

C $\int_a^b f'(x) \, dx$

D $\pi \int_a^b (f(x))^2 \, dx$

Exercise 3.1

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D $\pi \int_a^b (f(x))^2 \, dx$

3.1 B

- arc length integrates $\sqrt{1 + (f')^2}$

Exercise 3.2 (a)

3.2 A curve is $y = \frac{2}{3}x^{3/2}$ for $0 \leq x \leq 3$.

(a) Set up the arc-length integral.

[2]

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(a) Set up the arc-length integral.

[2]

(a)

- $f'(x) = x^{1/2}$, so $L = \int_0^3 \sqrt{1+x} \, dx$

Exercise 3.2 (b)

3.2 A curve is $y = \frac{2}{3}x^{3/2}$ for $0 \leq x \leq 3$.

(b) Evaluate it.

[3]

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(b) Evaluate it.

[3]

(b)

- $\left[\frac{2}{3}(1+x)^{3/2} \right]_0^3 = \frac{2}{3}(8-1) = \frac{14}{3}$

Exercise 3.3

3.3 Set up the integral for the length of $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$. (Use $\frac{d}{dx} \ln(\cos x) = -\tan x$ and $1 + \tan^2 x = \sec^2 x$.) [3]

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3.3

- $f'(x) = -\tan x$, so $1 + (f')^2 = 1 + \tan^2 x = \sec^2 x$; $L = \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx = \int_0^{\pi/4} \sec x \, dx$