Exercise walk-through

8.12
Volume with Washer Method: Revolving Around Other Axes

AP Calculus BC

Volumes with known cross sections

If a solid has a known **cross section** perpendicular to an axis, the volume is the integral of its area:

$$V = \int_a^b A(x)\,dx$$

The exam phrase "the base of a solid… cross sections are squares" signals exactly this.

square of side s: $A = s^2$

rectangle, height ks: $A = k\,s^2$

equilateral **triangle**: $A = \frac{\sqrt{3}}{4}s^2$

semicircle, diameter s: $A = \frac{\pi}{8}s^2$

Set the shape's key length equal to the slice length $s(x) = f(x) - g(x)$, then integrate.

Disc, washer and shell

Disc method	Rotating a region that touches the axis: $V = \pi \int R(x)^2\,dx$ – each slice is a solid circle.
Washer method	The **washer method** is for a region with a gap: $V = \pi \int \left(R^2 - r^2\right) dx$, outer radius minus inner.
Which variable	Slice perpendicular to the axis of rotation. Rotating about a horizontal axis integrates in x; a vertical axis, in y.
A shifted axis	Rotating about $y = k$ changes the radii to $\lvert f(x) - k \rvert$ – the method is unchanged.

Sketch the slice before writing the integral. Radius, thickness and limits all come from the picture.

Exam tips

- Area between curves is $\int(\text{top} - \text{bottom})\,dx$ – find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f\,dx$.
- Accumulated change is $\int$ of a rate: total $=$ initial value $+ \int_a^b(\text{rate})\,dt$.
- Integration means "adding up infinitely many tiny pieces" – set up the integrand as one thin slice.

Exercises 2.1–2.3

2.1 A curve $y = f(x)$ is revolved about $y = -2$. Write its radius. [1]

2.2 For outer curve $y = 4$ and inner curve $y = 1$ revolved about $y = -1$, state both radii. [2]

2.3 Expand $(x+1)^2 - (\sqrt{x}+1)^2$? No – first expand $(x+1)^2$. [1]

Exercises 2.1–2.3

2.1 A curve $y = f(x)$ is revolved about $y = -2$. Write its radius. [1]

2.1

- $R = f(x) + 2$

2.2 For outer curve $y = 4$ and inner curve $y = 1$ revolved about $y = -1$, state both radii. [2]

2.2

- outer $= 4 - (-1) = 5$; inner $= 1 - (-1) = 2$

2.3 Expand $(x+1)^2 - (\sqrt{x}+1)^2$? No – first expand $(x+1)^2$. [1]

2.3

- $(x+1)^2 = x^2 + 2x + 1$

Exercise 3.1

3.1 Revolving a region between two curves about $y = k$ uses radii [1]

- [ ] **A** the curves themselves
- [ ] **B** each curve's distance to the line $y = k$
- [ ] **C** only the outer curve
- [ ] **D** k for both

Exercise 3.1

3.1 Revolving a region between two curves about $y = k$ uses radii [1]

- [ ] **A** the curves themselves
- [ ] **B** each curve's distance to the line $y = k$
- [ ] **C** only the outer curve
- [ ] **D** k for both

3.1 B

- each radius is the curve's distance to the line $y = k$

Exercise 3.2 (a)

3.2 The region between $y = x^2$ (lower) and $y = 2x$ (upper) on $[0, 2]$ is revolved about the line $y = -1$.

(a) Write the outer and inner radii. [2]

Exercise 3.2 (a)

3.2 The region between $y = x^2$ (lower) and $y = 2x$ (upper) on $[0, 2]$ is revolved about the line $y = -1$.

(a) Write the outer and inner radii. [2]

(a)

- On $(0, 2)$, $2x \geq x^2$, so outer (from $2x$) $= 2x + 1$, inner (from x^2) $= x^2 + 1$

Exercise 3.2 (b)

3.2 The region between $y = x^2$ (lower) and $y = 2x$ (upper) on $[0, 2]$ is revolved about the line $y = -1$.

(b) Set up the volume integral. [2]

Exercise 3.2 (b)

3.2 The region between $y = x^2$ (lower) and $y = 2x$ (upper) on $[0, 2]$ is revolved about the line $y = -1$.

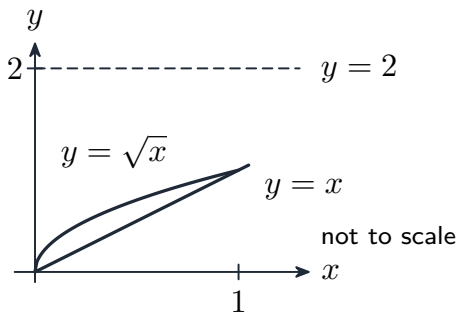
(b) Set up the volume integral. [2]

(b)

- $V = \pi \displaystyle\int_0^2 \left((2x+1)^2 - (x^2+1)^2\right) dx$

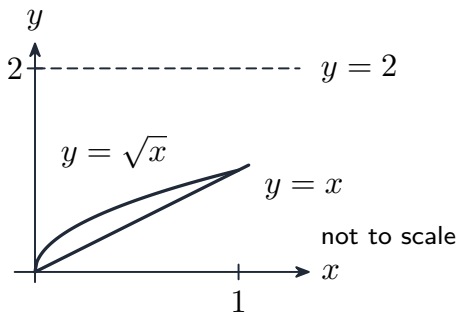
Exercise 3.3

3.3 The region between $y = \sqrt{x}$ and $y = x$ on $[0, 1]$ is revolved about the line $y = 2$. Write the outer and inner radii (as distances to $y = 2$), and set up the volume integral. [4]



Exercise 3.3

3.3 The region between $y = \sqrt{x}$ and $y = x$ on $[0, 1]$ is revolved about the line $y = 2$. Write the outer and inner radii (as distances to $y = 2$), and set up the volume integral. [4]



3.3

- Distances to $y = 2$: outer (from $y = x$, the lower curve, farther below 2) $= 2 - x$; inner (from $y = \sqrt{x}$) $= 2 - \sqrt{x}$; $V = \pi \int_0^1 \left((2 - x)^2 - (2 - \sqrt{x})^2\right) dx$