

Exercise walk-through

7.8

Exponential Models with Differential Equations

AP Calculus BC

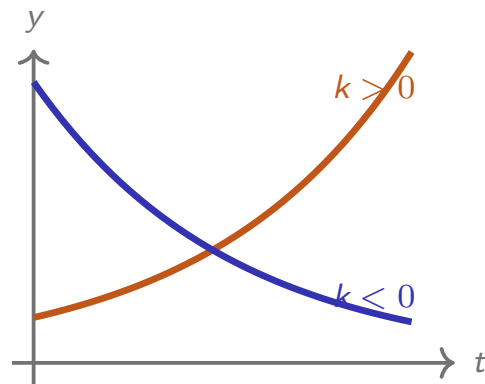
Exponential growth and decay

The most important model: rate
proportional to size.

$$\frac{dy}{dt} = ky \implies y = y_0 e^{kt}$$

where y_0 is the value at $t = 0$.

$k > 0$ gives **growth** ; $k < 0$ gives
decay .



The exponential model, and where it fails

The equation

Exponential growth or decay is the solution of $\frac{dy}{dt} = ky$: rate proportional to amount, giving $y = y_0 e^{kt}$.

Reading k

$k > 0$ grows, $k < 0$ decays, and $\ln 2/|k|$ is the doubling or half-life.

Where it fails

Nothing grows proportionally forever. The logistic model adds the factor $(1 - y/L)$, so growth slows as y approaches the carrying capacity L .

The tell

Fastest growth in a logistic model happens at $y = L/2$ – the inflection point of the curve.

Read the sentence into the differential equation: “proportional to the amount present” is $dy/dt = ky$.

Worked example – radioactive decay and half-life

$$\frac{dy}{dt} = -0.10 y \text{ (years)}, y_0 = 50 \text{ g}$$

The solution is $y = 50e^{-0.10t}$, so after 10 years

$$y = 50e^{-1} = 18.4 \text{ g.}$$

Half-life : solve $25 = 50e^{-0.10t}$, i.e.
 $e^{-0.10t} = \frac{1}{2}$:

$$t = \frac{\ln 2}{0.10} = 6.9 \text{ years.}$$

The half-life is independent of the starting amount – that is what makes it a useful constant.

Exercises 2.1·2.3

2.1 Write the solution of $\frac{dy}{dt} = 0.2y$ with $y(0) = 30$. [1]

2.2 A sample decays as $y = 80e^{-0.05t}$. Find the amount at $t = 10$. [2]

2.3 A population doubles: $y = y_0e^{kt}$ with $y_0 = 100$ and $y(2) = 200$. Find k . [3]

Exercises 2.1-2.3

2.1 Write the solution of $\frac{dy}{dt} = 0.2y$ with $y(0) = 30$. [1]

2.1

- $y = 30e^{0.2t}$

2.2 A sample decays as $y = 80e^{-0.05t}$. Find the amount at $t = 10$. [2]

2.2

- $80e^{-0.5} = 80(0.6065) \approx 48.5$

2.3 A population doubles: $y = y_0e^{kt}$ with $y_0 = 100$ and $y(2) = 200$. Find k . [3]

2.3

- $200 = 100e^{2k}$; $e^{2k} = 2$; $k = \frac{1}{2} \ln 2 \approx 0.347$

Exercise 3.1

3.1 The solution of $\frac{dy}{dt} = ky$ with $y(0) = y_0$ is

[1]

☐

A $y = y_0 + kt$

B $y = y_0 e^{kt}$

C $y = kt^2$

D $y = y_0 k^t$

Exercise 3.1

3.1 The solution of $\frac{dy}{dt} = ky$ with $y(0) = y_0$ is

[1]

☐

A $y = y_0 + kt$

B $y = y_0 e^{kt}$

C $y = kt^2$

D $y = y_0 k^t$

3.1 B

- $\frac{dy}{dt} = ky$ gives $y = y_0 e^{kt}$

Exercise 3.2 (a)

3.2 A radioactive sample follows $\frac{dA}{dt} = kA$, starting at $A(0) = 40$ mg, and 30 mg remains after 5 years.

(a) Find k .

[3]

Exercise 3.2 (a)

3.2 A radioactive sample follows $\frac{dA}{dt} = kA$, starting at $A(0) = 40$ mg, and 30 mg remains after 5 years.

(a) Find k .

[3]

(a)

- $30 = 40e^{5k}$; $e^{5k} = 0.75$; $k = \frac{1}{5} \ln 0.75 \approx -0.0575$

Exercise 3.2 (b)

3.2 A radioactive sample follows $\frac{dA}{dt} = kA$, starting at $A(0) = 40$ mg, and 30 mg remains after 5 years.

(b) Predict the amount after 12 years.

[2]

Exercise 3.2 (b)

3.2 A radioactive sample follows $\frac{dA}{dt} = kA$, starting at $A(0) = 40$ mg, and 30 mg remains after 5 years.

(b) Predict the amount after 12 years.

[2]

(b)

- $A = 40e^{-0.0575(12)} = 40e^{-0.690} \approx 20.0$ mg

Exercise 3.3 (a)

3.3 A bacteria count grows exponentially from 1000 to 8000 in 6 hours.

(a) Find the growth constant k .

[2]

Exercise 3.3 (a)

3.3 A bacteria count grows exponentially from 1000 to 8000 in 6 hours.

(a) Find the growth constant k .

[2]

(a)

- $8000 = 1000e^{6k} \Rightarrow e^{6k} = 8 \Rightarrow k = \frac{1}{6} \ln 8 \approx 0.347$

Exercise 3.3 (b)

3.3 A bacteria count grows exponentially from 1000 to 8000 in 6 hours.

(b) How long, from the start, until the count reaches 16 000?

[3]

Exercise 3.3 (b)

3.3 A bacteria count grows exponentially from 1000 to 8000 in 6 hours.

(b) How long, from the start, until the count reaches 16 000?

[3]

(b)

- $$16000 = 1000e^{kt} \Rightarrow e^{kt} = 16 \Rightarrow t = \frac{\ln 16}{0.347} \approx 8 \text{ h}$$