

Exercise walk-through

7.6

Finding General Solutions Using Separation of Variables

AP Calculus BC

Separation of variables

If $\frac{dy}{dx}$ factors into a function of x times a function of y , split them and integrate:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x) dx.$$

Add the constant of integration **once**
– on the x -side.

all y one side,
all x the other



integrate both sides



the **general so-**
lution

Worked example – separate, integrate, then pin it down

Solve $\frac{dy}{dx} = xy$ **with** $y(0) = 2$

$$\int \frac{dy}{y} = \int x dx \Rightarrow \ln |y| = \frac{x^2}{2} + C$$

$$\Rightarrow y = Ae^{x^2/2}$$

Applying $y(0) = 2$ gives $A = 2$:

$$y = 2e^{x^2/2}.$$

An **initial condition**

(x_0, y_0) picks **one** curve from the family – there is exactly **one** solution through a given point.

From the family to the particular solution

The constant gives a **family** of curves; the initial condition selects the **particular solution** .

Watch **domain restrictions** : keep the **branch containing the initial point**
(e.g. the right sign of a square root).

Euler's method, and what it is good for

the idea

step along the slope field: $y_{n+1} = y_n + f(x_n, y_n) \Delta x$

the exam skill

carry out **two or three steps by hand** from a table – that is what is asked

the tabulation

columns for x , y , the slope, and Δy ; one row per step

what it produces

an **approximation**, not a solution

the error

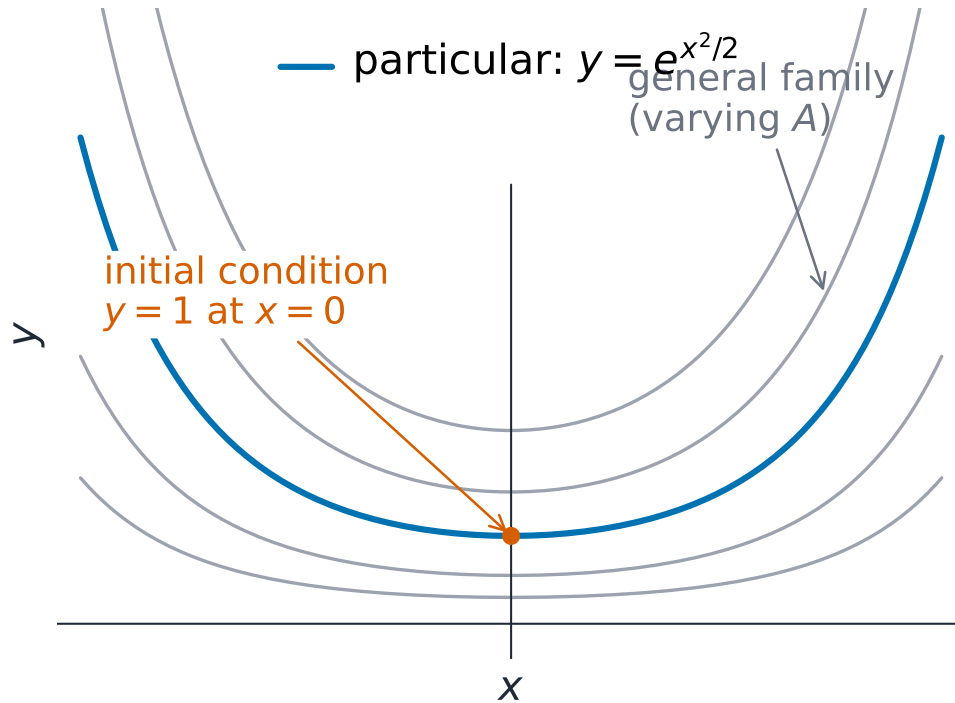
Euler's method **underestimates** a concave-up solution and overestimates a concave-down one

smaller steps

reduce the error, at the cost of more arithmetic

The concavity argument is the same one as for a tangent-line approximation – and for the same reason: both follow a straight line away from a curving one.

From the family to the particular solution, in detail



The constant gives a **family** of curves; the initial condition selects the **particular solution**.

Exercises 2.1-2.3

2.1 Separate the variables in $\frac{dy}{dx} = 2xy$. [1]

2.2 Solve $\frac{dy}{dx} = \frac{1}{y}$ (find the general solution). [3]

2.3 Solve $\frac{dy}{dx} = 3y$ (general solution in terms of a constant A). [3]

Exercises 2.1-2.3

2.1 Separate the variables in $\frac{dy}{dx} = 2xy$. [1]

2.1

- $\frac{1}{y} dy = 2x dx$

2.2 Solve $\frac{dy}{dx} = \frac{1}{y}$ (find the general solution). [3]

2.2

- $y dy = dx; \frac{1}{2}y^2 = x + C; y^2 = 2x + C_1$

2.3 Solve $\frac{dy}{dx} = 3y$ (general solution in terms of a constant A). [3]

2.3

- $\frac{1}{y} dy = 3 dx; \ln |y| = 3x + C; y = Ae^{3x}$

Exercise 3.1

3.1 After separating $\frac{dy}{dx} = ky$, integrating the left side gives

[1]

☐**A** y **B** $\ln|y|$ **C** $\frac{1}{y^2}$ **D** e^y

Exercise 3.1

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[1]

☐**A** y **B** $\ln |y|$ **C** $\frac{1}{y^2}$ **D** e^y

3.1 B

- $\int \frac{1}{y} dy = \ln |y|$

Exercise 3.2 (a)

3.2 Solve $\frac{dy}{dx} = xy^2$.

(a) Separate and integrate both sides.

[3]

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3.2 Solve $\frac{dy}{dx} = xy^2$.

(a) Separate and integrate both sides.

[3]

(a)

- $\frac{1}{y^2} dy = x dx; -\frac{1}{y} = \frac{x^2}{2} + C$

Exercise 3.2 (b)

3.2 Solve $\frac{dy}{dx} = xy^2$.

(b) Solve for y in terms of x and a constant.

[2]

Exercise 3.2 (b)

3.2 Solve $\frac{dy}{dx} = xy^2$.

(b) Solve for y in terms of x and a constant.

[2]

(b)

- $$y = -\frac{1}{\frac{x^2}{2} + C} = \frac{-2}{x^2 + C_1}$$

Exercise 3.3

3.3 Solve $\frac{dy}{dx} = \frac{\cos x}{y}$, giving the general solution.

[3]

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[3]

3.3

- $y \, dy = \cos x \, dx; \frac{1}{2}y^2 = \sin x + C; y^2 = 2 \sin x + C_1$