

Exercise walk-through

5.10

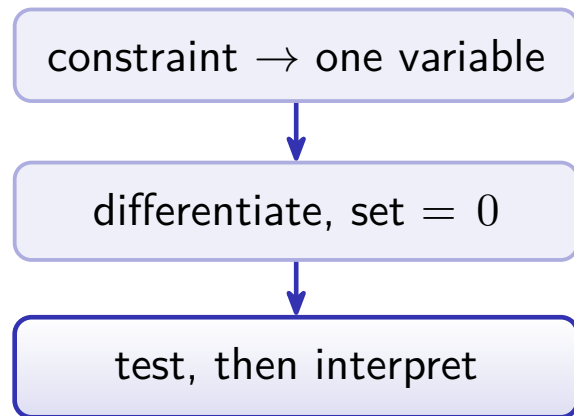
Introduction to Optimization Problems

AP Calculus BC

Solving an optimization problem

Optimization is the same machinery aimed at a real goal:

- 1 write the quantity as a function of **one** variable (use the constraint to eliminate the rest)
- 2 state the allowed interval
- 3 find and **test** the critical points
- 4 answer with **units** and interpretation



Worked example – the farmer's fence

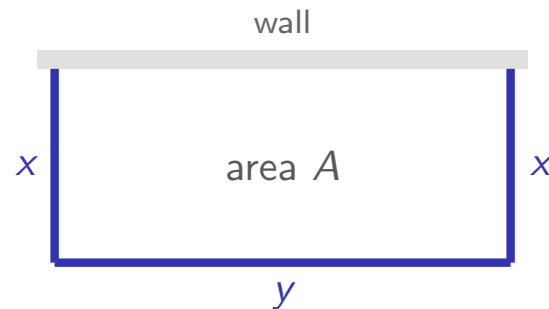
100 m of fence, three sides (a wall is the fourth). Maximize the area.

Ends x , side y : constraint $2x + y = 100$, so $y = 100 - 2x$.

$$A(x) = x(100 - 2x) = 100x - 2x^2$$

$$A'(x) = 100 - 4x = 0 \Rightarrow x = 25$$

$A'' = -4 < 0$, a maximum: $y = 50$ and $A = 1250 \text{ m}^2$.



Behaviour of implicit relations

It all extends to **implicitly** defined relations:

- horizontal tangent where $\frac{dy}{dx} = 0$
- vertical tangent where $\frac{dy}{dx}$ is **undefined**

$\frac{dy}{dx}$ usually involves **both** x and y – substitute it back in when finding $\frac{d^2y}{dx^2}$, then read concavity from the sign.

Exercises 2.1·2.3

2.1 A rectangle has perimeter 24 cm. Write its **area** as a function of its width x . [2]

2.2 An open-top box has a square base of side x and volume 32 cm^3 . Write its **surface area** (base + 4 sides) as a function of x . [2]

2.3 For the farmer's pen $A(y) = 40y - 2y^2$, state the domain of y . [1]

Exercises 2.1-2.3

2.1 A rectangle has perimeter 24 cm. Write its **area** as a function of its width x . [2]

2.1

- Perimeter $2x + 2w = 24 \Rightarrow \text{length} = 12 - x$; $A = x(12 - x) = 12x - x^2$

2.2 An open-top box has a square base of side x and volume 32 cm^3 . Write its **surface area** (base + 4 sides) as a function of x . [2]

2.2

- Height from $x^2h = 32$, so $h = 32/x^2$; $S = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$

2.3 For the farmer's pen $A(y) = 40y - 2y^2$, state the domain of y . [1]

2.3

- $0 < y < 20$

Exercise 3.1

3.1 A quantity Q is to be maximized subject to a constraint. What is the **first** step? [1]

☐

A differentiate Q immediately

B write Q as a function of a single variable using the constraint

C set $Q = 0$

D find the second derivative

Exercise 3.1

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A differentiate Q immediately

B write Q as a function of a single variable using the constraint

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3.1 B

- reduce to a single variable using the constraint before differentiating

Exercise 3.2 (a)

3.2 A cylindrical can has volume 355 cm^3 . Its surface area is $S = 2\pi r^2 + 2\pi rh$.

(a) Use the volume constraint $\pi r^2 h = 355$ to write S as a function of r only. [2]

Exercise 3.2 (a)

3.2 A cylindrical can has volume 355 cm^3 . Its surface area is $S = 2\pi r^2 + 2\pi rh$.

(a) Use the volume constraint $\pi r^2 h = 355$ to write S as a function of r only. [2]

(a)

- $$h = \frac{355}{\pi r^2}; S(r) = 2\pi r^2 + 2\pi r \left(\frac{355}{\pi r^2} \right) = 2\pi r^2 + \frac{710}{r}$$

Exercise 3.2 (b)

3.2 A cylindrical can has volume 355 cm^3 . Its surface area is $S = 2\pi r^2 + 2\pi rh$.

(b) State the domain of r .

[1]

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(b) State the domain of r .

[1]

(b)

- $r > 0$

Exercise 3.3

3.3 A 600 cm^2 sheet of card is folded into an open box with a square base of side x and height h , using all the card ($x^2 + 4xh = 600$). Write the **volume** V as a function of x alone. [3]

Exercise 3.3

3.3 A 600 cm^2 sheet of card is folded into an open box with a square base of side x and height h , using all the card ($x^2 + 4xh = 600$). Write the **volume** V as a function of x alone. [3]

3.3

$$\bullet \quad h = \frac{600 - x^2}{4x}; \quad V = x^2h = x^2 \left(\frac{600 - x^2}{4x} \right) = \frac{600x - x^3}{4} = 150x - \frac{1}{4}x^3$$