

Exercise walk-through

10.6

Comparison Tests for Convergence

AP Calculus BC

Start here – the n th term test

If the terms do not shrink to zero, the sum cannot settle:

$$\lim_{n \rightarrow \infty} a_n \neq 0 \implies \text{diverges.}$$

Always check this **quick test first**.

✗ It is **only** a test for **divergence**

If $a_n \rightarrow 0$ the test is **inconclusive** – the series may **still** diverge

p -series and the harmonic series

$$\sum \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

The **integral test** proves this:
for positive, decreasing, continuous f ,
 $\sum a_n$ and $\int_1^\infty f$ **both** converge or **both** diverge.

The **harmonic series** $\sum \frac{1}{n}$
($p = 1$) **diverges** – even though its
terms go to zero. A famous, must-
know fact.

The p -series family is the standard **yardstick**
for the comparison tests.

Comparison and alternating series

Direct comparison

$0 \leq a_n \leq b_n$ and $\sum b_n$
converges $\Rightarrow$ so does $\sum a_n$

Limit comparison

$\lim \frac{a_n}{b_n}$ finite and positive
 $\Rightarrow$ both do the **same** thing

Alternating series

b_n positive, **decreasing**,
and $b_n \rightarrow 0 \Rightarrow$ converges

Ratio test

$L = \lim \left| \frac{a_{n+1}}{a_n} \right|$: $L < 1$ converges,
 $L > 1$ diverges, $L = 1$ **inconclusive**

Worked example – the ratio test

Test $\sum \frac{n}{2^n}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \rightarrow \frac{1}{2} < 1$$

so the series **converges**.

The ratio test is the go-to for **factorials** or **n th powers** – and it is exactly how you find a **radius of convergence**.

Absolute vs conditional convergence

Absolutely

$\sum |a_n|$ converges –
the **stronger** property

Conditionally

$\sum a_n$ converges but
 $\sum |a_n|$ does not

The classic: $\sum \frac{(-1)^n}{n}$ converges **conditionally** – it relies on the **cancellation of signs**, since $\sum \frac{1}{n}$ diverges.

Alternating error bound

$$|S - S_N| \leq b_{N+1}$$

The error is **no larger than the first omitted term**.

Exercises 2.1·2.3

2.1 For $\sum \frac{1}{n^3 + 2}$, name a good comparison series.

[1]

2.2 Is $\frac{1}{n^3 + 2} \leq \frac{1}{n^3}$? What does this tell you?

[2]

2.3 For limit comparison of $\sum \frac{n}{n^2 + 1}$, a good b_n is $\frac{1}{n}$. Find $\lim \frac{a_n}{b_n}$.

[2]

Exercises 2.1·2.3

2.1 For $\sum \frac{1}{n^3 + 2}$, name a good comparison series. [1]

2.1

- $\sum \frac{1}{n^3}$ (a convergent p-series, $p = 3$)

2.2 Is $\frac{1}{n^3 + 2} \leq \frac{1}{n^3}$? What does this tell you? [2]

2.2

- Yes; since the larger series $\sum \frac{1}{n^3}$ converges, so does $\sum \frac{1}{n^3 + 2}$

2.3 For limit comparison of $\sum \frac{n}{n^2 + 1}$, a good b_n is $\frac{1}{n}$. Find $\lim \frac{a_n}{b_n}$. [2]

2.3

- $\frac{a_n}{b_n} = \frac{n/(n^2 + 1)}{1/n} = \frac{n^2}{n^2 + 1} \rightarrow 1$

Exercises 3.1·3.2

3.1 In the limit comparison test, if $\lim \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then [1]

- ☐ **A** $\sum a_n$ always converges
B $\sum a_n$ and $\sum b_n$ behave the same way
C $\sum a_n$ always diverges
D the test is inconclusive

3.2 Use direct comparison to show $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ converges. [3]

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3.1 B

- they converge or diverge together

3.2 Use direct comparison to show $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ converges. [3]

3.2

- $\frac{1}{2^n + 1} \leq \frac{1}{2^n}$; $\sum \left(\frac{1}{2}\right)^n$ is geometric with $r = \frac{1}{2} < 1$, converges; so by direct comparison the series converges

Exercise 3.3

3.3 Use limit comparison with $b_n = \frac{1}{n}$ to decide whether $\sum \frac{n+2}{n^2+3}$ converges. [4]

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3.3

- $\frac{a_n}{b_n} = \frac{(n+2)n}{n^2+3} = \frac{n^2+2n}{n^2+3} \rightarrow 1$; finite positive, and $\sum \frac{1}{n}$ diverges, so the series **diverges**