

Exercise walk-through

10.5

Harmonic Series and p-Series

AP Calculus BC

p -series and the harmonic series

$$\sum \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

The **integral test** proves this:
for positive, decreasing, continuous f ,
 $\sum a_n$ and $\int_1^\infty f$ **both** converge or **both** diverge.

The **harmonic series** $\sum \frac{1}{n}$
($p = 1$) **diverges** – even though its
terms go to zero. A famous, must-
know fact.

The p -series family is the standard **yardstick**
for the comparison tests.

Absolute against conditional convergence

Converges absolutely

A series **converges absolutely** when $\sum |a_n|$ converges – and then $\sum a_n$ converges too, whatever the signs do.

Converges conditionally

It **converges conditionally** when $\sum a_n$ converges but $\sum |a_n|$ does not – the alternating harmonic series is the standard case.

Why it matters

An absolutely convergent series can be rearranged freely; a conditionally convergent one can be rearranged to give *any* sum at all.

How to test

Test $\sum |a_n|$ first. If it converges you are finished; if not, try the alternating series test on $\sum a_n$.

“Converges” is not one property. Say which kind, because the error bound and the rearrangement both depend on it.

Exam tips

- Test a series for convergence with the right tool: **geometric** ($|r| < 1$, sum $\frac{a}{1-r}$), n th-term, ratio, integral, comparison, or alternating-series test.
- A **geometric** infinite sum converges only when $|r| < 1$; otherwise it diverges.
- Build a **Taylor/Maclaurin** series to approximate a function; more terms give a better fit **near the centre**.
- Know the standard Maclaurin series for e^x , $\sin x$, $\cos x$, and $\frac{1}{1-x}$.
- Find the **radius/interval of convergence** with the ratio test, then check the endpoints separately.

Exercises 2.1-2.3

2.1 State whether $\sum \frac{1}{n^4}$ converges.

[1]

2.2 State whether $\sum \frac{1}{n^{2/3}}$ converges.

[1]

2.3 Does the harmonic series converge? Give the value of p .

[2]

Exercises 2.1·2.3

2.1 State whether $\sum \frac{1}{n^4}$ converges. [1]

2.1

- Converges ($p = 4 > 1$)

2.2 State whether $\sum \frac{1}{n^{2/3}}$ converges. [1]

2.2

- Diverges ($p = \frac{2}{3} \leq 1$)

2.3 Does the harmonic series converge? Give the value of p . [2]

2.3

- No, it diverges; $p = 1$

Exercise 3.1

3.1 $\sum \frac{1}{n^p}$ converges if and only if

[1]

☐

A $p < 1$

B $p = 1$

C $p > 1$

D $p \geq 0$

Exercise 3.1

3.1 $\sum \frac{1}{n^p}$ converges if and only if

[1]



A $p < 1$

B $p = 1$

C $p > 1$

D $p \geq 0$

3.1 C

- converges iff $p > 1$

Exercise 3.2 (a)

3.2 Classify each as convergent or divergent, stating p :

(a) $\sum \frac{1}{n^{1.2}}$

[1]

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3.2 Classify each as convergent or divergent, stating p :

(a) $\sum \frac{1}{n^{1.2}}$

[1]

(a)

- convergent, $p = 1.2 > 1$

Exercise 3.2 (b)

3.2 Classify each as convergent or divergent, stating p :

(b) $\sum \frac{1}{n^{0.9}}$

[1]

Exercise 3.2 (b)

3.2 Classify each as convergent or divergent, stating p :

(b) $\sum \frac{1}{n^{0.9}}$

[1]

(b)

- divergent, $p = 0.9 \leq 1$

Exercise 3.2 (c)

3.2 Classify each as convergent or divergent, stating p :

(c) $\sum \frac{1}{n}$

[1]

Exercise 3.2 (c)

3.2 Classify each as convergent or divergent, stating p :

(c) $\sum \frac{1}{n}$

[1]

(c)

- divergent, $p = 1$

Exercise 3.3

3.3 Explain why $\sum \frac{1}{\sqrt{n}}$ diverges but $\sum \frac{1}{n^2}$ converges, in terms of p .

[2]

Exercise 3.3

3.3 Explain why $\sum \frac{1}{\sqrt{n}}$ diverges but $\sum \frac{1}{n^2}$ converges, in terms of p . [2]

3.3

- $\sum_{p=1} \frac{1}{\sqrt{n}}$ has $p = \frac{1}{2} \leq 1$ (diverges); $\sum \frac{1}{n^2}$ has $p = 2 > 1$ (converges) – the boundary is