

Exercise walk-through

# 10.15

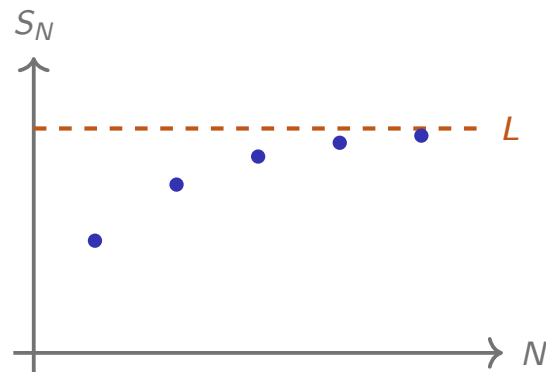
## Representing Functions as Power Series

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AP Calculus BC

# What we will cover

- 1 Convergence & partial sums
- 2 Geometric series
- 3 The convergence tests
- 4 Absolute vs conditional
- 5 Taylor polynomials & error
- 6 Power series & intervals



# Geometric series – the one you can sum exactly

A **geometric series** has a constant ratio  $r$ . It converges **exactly** when  $|r| < 1$ :

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

It underlies **power series** later in the unit.

**Sum**  $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \dots$

Here  $a = 3$  and  $r = \frac{1}{2}$ , and  $|r| < 1$ :

$$\frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = 6.$$

# Radius and interval of convergence

A **power series**  $\sum c_n(x - a)^n$  converges within a **radius**  $R$  of its centre. Find  $R$  with the **ratio test**, then test the **endpoints separately**.

The ratio test is **inconclusive at the endpoints** – that is why they need their own check.

**Interval of**  $\sum \frac{x^n}{n}$

Ratio:  $|x| \frac{n}{n+1} \rightarrow |x| < 1$ , so  $R = 1$ .  
 $x = -1$ : alternating harmonic – **converges**.  
 $x = 1$ : harmonic – **diverges**.

Interval =  $[-1, 1)$ .

## Exercises 2.1·2.2

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**2.1** Differentiate the series  $\sum_{n=0}^{\infty} x^n$  term by term. [2]

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**2.2** Write the Maclaurin series for  $\frac{1}{1-x}$  and integrate its first three terms. [2]

## Exercises 2.1·2.2

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**2.1** Differentiate the series  $\sum_{n=0}^{\infty} x^n$  term by term. [2]

**2.1**

- $\frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1}$

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**2.2** Write the Maclaurin series for  $\frac{1}{1-x}$  and integrate its first three terms. [2]

**2.2**

- $\sum x^n = 1 + x + x^2 + \cdots$ ; integrating:  $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

## Exercises 2.3-3.1

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**2.3** State the first three nonzero terms of the series for  $\arctan x$ .

[1]

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**3.1** Within its radius of convergence, a power series may be

[1]

☐

**A** differentiated but not integrated

**B** integrated but not differentiated

**C** both differentiated and integrated term by term

**D** neither

## Exercises 2.3·3.1

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**2.3** State the first three nonzero terms of the series for  $\arctan x$ .

[1]

**2.3**

- $x - \frac{x^3}{3} + \frac{x^5}{5}$

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**3.1** Within its radius of convergence, a power series may be

[1]

☐

A differentiated but not integrated

B integrated but not differentiated

C both differentiated and integrated term by term

D neither

**3.1 C**

- both, term by term, within the radius

## Exercise 3.2 (a)

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**3.2** Start from  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

(a) Write the series for  $e^{-x^2}$ .

[2]

## Exercise 3.2 (a)

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**3.2** Start from  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

(a) Write the series for  $e^{-x^2}$ .

[2]

(a)

$$\bullet \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \dots$$

## Exercise 3.2 (b)

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**3.2** Start from  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

(b) Write the first three terms of  $\int_0^{\cdot} e^{-x^2} dx$  obtained by integrating that series term by term.

[3]

## Exercise 3.2 (b)

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**3.2** Start from  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

(b) Write the first three terms of  $\int_0^1 e^{-x^2} dx$  obtained by integrating that series term by term. [3]

(b)

$$\bullet \int_0^1 (1 - x^2 + \frac{x^4}{2} - \dots) dx = \left[ x - \frac{x^3}{3} + \frac{x^5}{10} - \dots \right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \dots$$

## Exercise 3.3

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**3.3** Use the geometric series to write a power series for  $\frac{1}{1-2x}$ , and state its radius of convergence. [3]

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**3.3**

- $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ ; converges for  $|2x| < 1$ , i.e.  $R = \frac{1}{2}$