

Exercise walk-through

10.14

Finding Taylor or Maclaurin Series for a Function

AP Calculus BC

Taylor polynomials

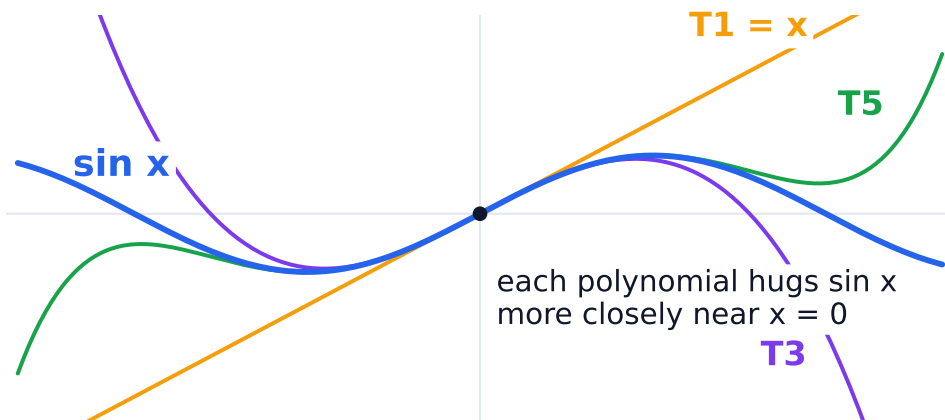
A **Taylor polynomial**
there:

approximates f near a centre a using its derivatives

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

Each added term matches **one more derivative**. Centred at $a = 0$ it is a **Maclaurin** polynomial.

Taylor polynomials, in detail



$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \quad \text{for}$$

$$\sin x: x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Each added term matches **one more derivative** at the centre, so the polynomial hugs the curve more closely – and only near a .

The Lagrange error bound

How far can a Taylor polynomial be from the truth?

$$|R_n(x)| \leq \frac{\max |f^{(n+1)}(z)|}{(n+1)!} |x - a|^{n+1}.$$

Bound the $(n+1)$ th derivative on the interval, then compute – the standard way to [prove](#) an estimate is accurate enough.

Radius and interval of convergence

A **power series** $\sum c_n(x - a)^n$ converges within a **radius** R of its centre. Find R with the **ratio test**, then test the **endpoints separately**.

The ratio test is **inconclusive at the endpoints** – that is why they need their own check.

Interval of $\sum \frac{x^n}{n}$

Ratio: $|x| \frac{n}{n+1} \rightarrow |x| < 1$, so $R = 1$.
 $x = -1$: alternating harmonic – **converges**.
 $x = 1$: harmonic – **diverges**.

Interval = $[-1, 1)$.

The four Maclaurin series to memorise

$$e^x = \sum \frac{x^n}{n!}$$

$$\frac{1}{1-x} = \sum x^n$$

$$\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

New series come from **manipulating** these – substituting, differentiating, integrating, multiplying.

Exercises 2.1·2.3

2.1 Write the Maclaurin series for e^x (summation form). [1]

2.2 Write the first three nonzero terms of $\cos x$. [2]

2.3 Use substitution to write the Maclaurin series for e^{-x} . [2]

Exercises 2.1·2.3

2.1 Write the Maclaurin series for e^x (summation form). [1]

2.1

$$\bullet \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2.2 Write the first three nonzero terms of $\cos x$. [2]

2.2

$$\bullet 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

2.3 Use substitution to write the Maclaurin series for e^{-x} . [2]

2.3

$$\bullet \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Exercises 3.1-3.2

3.1 The Maclaurin series for $\frac{1}{1-x}$ is

[1]

☐

A $\sum (-1)^n x^n$

B $\sum x^n$

C $\sum \frac{x^n}{n!}$

D $\sum nx^n$

3.2 Find the Maclaurin series for $\frac{1}{1+x^2}$ (substitute into the geometric series).

[3]

Exercises 3.1·3.2

3.1 The Maclaurin series for $\frac{1}{1-x}$ is

[1]

- ☐ A $\sum (-1)^n x^n$
☐ B $\sum x^n$
☐ C $\sum \frac{x^n}{n!}$
☐ D $\sum nx^n$

3.1 B

• $\frac{1}{1-x} = \sum x^n$

3.2 Find the Maclaurin series for $\frac{1}{1+x^2}$ (substitute into the geometric series).

[3]

3.2

• $\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

Exercise 3.3

3.3 Find the first three nonzero terms of the Maclaurin series for $x \sin x$.

[3]

Exercise 3.3

3.3 Find the first three nonzero terms of the Maclaurin series for $x \sin x$.

[3]

3.3

- $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$; multiply by x : $x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \cdots$