

Exercise walk-through

10.11

Finding Taylor Polynomial Approximations of Functions

AP Calculus BC

Taylor polynomials

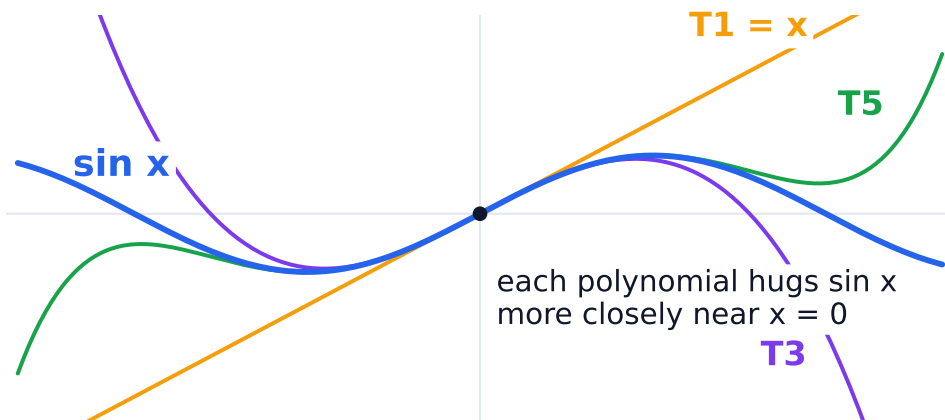
A **Taylor polynomial**
there:

approximates f near a centre a using its derivatives

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

Each added term matches **one more derivative**. Centred at $a = 0$ it is a **Maclaurin** polynomial.

Taylor polynomials, in detail



$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \quad \cdot \text{ for}$$

$$\sin x: x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Each added term matches **one more derivative** at the centre, so the polynomial hugs the curve more closely – and only near a .

The Lagrange error bound

How far can a Taylor polynomial be from the truth?

$$|R_n(x)| \leq \frac{\max |f^{(n+1)}(z)|}{(n+1)!} |x - a|^{n+1}.$$

Bound the $(n+1)$ th derivative on the interval, then compute – the standard way to **prove** an estimate is accurate enough.

Radius and interval of convergence

A **power series** $\sum c_n(x - a)^n$ converges within a **radius** R of its centre. Find R with the **ratio test**, then test the **endpoints separately**.

The ratio test is **inconclusive at the endpoints** – that is why they need their own check.

Interval of $\sum \frac{x^n}{n}$

Ratio: $|x| \frac{n}{n+1} \rightarrow |x| < 1$, so $R = 1$.
 $x = -1$: alternating harmonic – **converges**.
 $x = 1$: harmonic – **diverges**.

Interval = $[-1, 1)$.

The four Maclaurin series to memorise

$$e^x = \sum \frac{x^n}{n!}$$

$$\frac{1}{1-x} = \sum x^n$$

$$\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

New series come from **manipulating** these – substituting, differentiating, integrating, multiplying.

Exercises 2.1·2.3

2.1 Write the general term $\frac{f^{(k)}(a)}{k!}(x-a)^k$ for $k = 2$. [1]

2.2 Given $f(0) = 1$, $f'(0) = 2$, $f''(0) = 6$, write $P_2(x)$ (Maclaurin). [2]

2.3 For $f(x) = e^x$, write the 2nd-degree Maclaurin polynomial. [2]

Exercises 2.1·2.3

2.1 Write the general term $\frac{f^{(k)}(a)}{k!}(x-a)^k$ for $k = 2$. [1]

2.1

- $\frac{f''(a)}{2!}(x-a)^2$

2.2 Given $f(0) = 1$, $f'(0) = 2$, $f''(0) = 6$, write $P_2(x)$ (Maclaurin). [2]

2.2

- $P_2(x) = 1 + 2x + \frac{6}{2}x^2 = 1 + 2x + 3x^2$

2.3 For $f(x) = e^x$, write the 2nd-degree Maclaurin polynomial. [2]

2.3

- $1 + x + \frac{x^2}{2}$

Exercise 3.1

3.1 The coefficient of $(x - a)^2$ in a Taylor polynomial is

[1]

☐

A $f''(a)$

B $\frac{f''(a)}{2}$

C $2f''(a)$

D $f'(a)$

Exercise 3.1

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3.1 B

- the coefficient is $\frac{f''(a)}{2!} = \frac{f''(a)}{2}$

Exercise 3.2 (a)

3.2 A function has $f(1) = 3$, $f'(1) = -2$, $f''(1) = 4$, $f'''(1) = 12$.

(a) Write the 3rd-degree Taylor polynomial about $x = 1$.

[3]

Exercise 3.2 (a)

3.2 A function has $f(1) = 3$, $f'(1) = -2$, $f''(1) = 4$, $f'''(1) = 12$.

(a) Write the 3rd-degree Taylor polynomial about $x = 1$. [3]

(a)

- $$P_3(x) = 3 - 2(x - 1) + \frac{4}{2}(x - 1)^2 + \frac{12}{6}(x - 1)^3 = 3 - 2(x - 1) + 2(x - 1)^2 + 2(x - 1)^3$$

Exercise 3.2 (b)

3.2 A function has $f(1) = 3$, $f'(1) = -2$, $f''(1) = 4$, $f'''(1) = 12$.

(b) Use it to estimate $f(1.1)$.

[2]

Exercise 3.2 (b)

3.2 A function has $f(1) = 3$, $f'(1) = -2$, $f''(1) = 4$, $f'''(1) = 12$.

(b) Use it to estimate $f(1.1)$.

[2]

(b)

- at $x = 1.1$: $3 - 0.2 + 2(0.01) + 2(0.001) = 2.822$

Exercise 3.3

3.3 Write the 4th-degree Maclaurin polynomial for $f(x) = \cos x$ (using $\cos 0 = 1$, and the derivative pattern). [3]

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3.3

- Derivatives of $\cos$ at 0: $1, 0, -1, 0, 1$; $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$