HOLIDAY HOMEWORK 假期作业

AP Calculus BC

Exercise sheets and past-paper practice, topic by topic.
每个单元：练习卷 + 历年真题。

For class or self-study • no answer keys included.

Contents 目录

1 What a limit is – Part 1

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
limit	极限	

Recall

You have looked at graphs and how they behave:

- You can trace a curve and see where it is heading.
- Some graphs get closer and closer to a value without quite reaching it.
- You know how to put a number into a formula.

This sheet introduces the **limit** – the first big idea of calculus. Each idea has a worked example.

New ideas

■ 1 What a limit is

A **limit** 极限 is the value a function **approaches** as x gets close to some point. We write it

$$\lim_{x\to a} f(x).$$

It asks: as x heads toward a, what does $f(x)$ head toward?

■ 2 Finding a limit

If the function has no break at a, you can just substitute.

Worked example. $\lim_{x\to 2}(x^2+1) = 2^2+1 = 5$.

■ 3 A hole in the graph

A limit can exist even if the function has a **hole** at that point – the graph still heads toward a value from both sides.

■ 4 From both sides

The limit exists only if the function heads toward the **same** value from the left and from the right.

Watch out: the limit is about what $f(x)$ **approaches**, not necessarily the value at that exact point. A function can approach 5 even if, right at that point, it has a hole or a different value.

Practice

Try these in order. The methods are all above.

2.1 In words, state what $\lim_{x\to a} f(x)$ means. [2]

2.2 Find $\lim_{x\to 3}(x+4)$. [2]

2.3 Find $\lim_{x\to 2}(x^2-1)$. [2]

2.4 State whether a limit can exist at a point where the graph has a hole. [1]

2.5 State the condition on the left and right sides for a limit to exist. [2]

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25+t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

1 Continuity and asymptotes – Part 2

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
continuous	连续	
vertical asymptote	渐近线	

Recall

In Part A you met limits. Now use them to describe how smooth a graph is:

- Some graphs you can draw without lifting your pen.
- Others have jumps, holes, or shoot off to infinity.

Each idea has a worked example before the Practice.

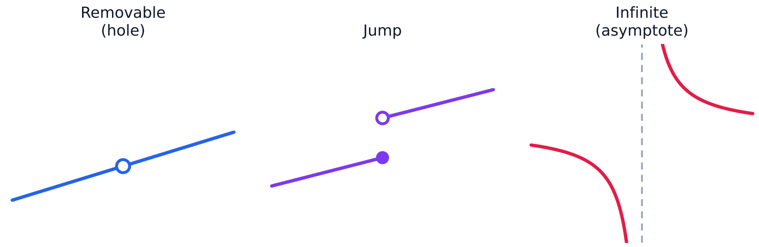
New ideas

■ 1 Continuity

A function is **continuous** 连续 at a point if it has no break there – no hole, jump, or gap. A continuous graph can be drawn without lifting your pen.

■ 2 Discontinuities

Where a graph breaks, it is **discontinuous**. The main kinds are a **hole**, a **jump**, and an **asymptote**.



f continuous at a iff $\lim_{x\to a} f(x)=f(a)$ (limit exists and equals the value)

■ 3 Vertical asymptotes

A **vertical asymptote** 渐近线 is a line the graph shoots up (or down) toward, without ever touching – it happens where the function blows up to infinity (like dividing by zero).

■ 4 Horizontal asymptotes

A **horizontal asymptote** is a value the graph levels off toward as x gets very large (or very negative).

Watch out: a graph can **cross** a horizontal asymptote but never crosses a **vertical** one – near a vertical asymptote the function races off to $\pm\infty$ instead.

Practice

Try these in order. The methods are all above.

2.1 State what it means for a function to be continuous at a point. [2]

2.2 Name three kinds of discontinuity. [3]

2.3 State what happens to a function near a vertical asymptote. [2]

2.4 State what a horizontal asymptote describes. [2]

2.5 For $f(x) = \dfrac{1}{x-3}$, state where the vertical asymptote is. [1]

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

2 The derivative as a slope – Part 1

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
derivative	导数	
tangent	切线	

Recall

You already know about the gradient (slope) of a line:

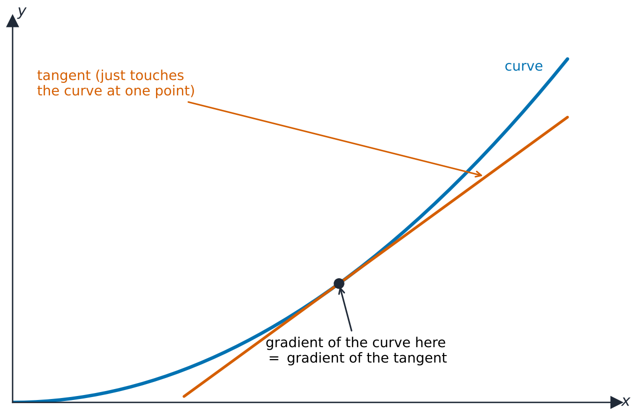
- A steep line has a big gradient; a flat one has zero.
- Gradient = rise over run.
- But a curve's steepness keeps changing along it.

This sheet introduces the **derivative** – the slope of a curve. Each idea has a worked example.

New ideas

■ 1 The slope of a curve

The **derivative** 导数 of a function at a point is the **slope of the curve** there – how steep it is at that exact spot.



■ 2 From secant to tangent

Take two points on the curve and join them – the **secant line**. Its slope is the **average** rate of change. As the two points move closer together, the secant becomes the **tangent** 切线 line, whose slope is the derivative.

■ 3 Notation

The derivative of f is written $f'(x)$ or $\dfrac{dy}{dx}$.

■ 4 A rate of change

The derivative is a **rate of change**: how fast the output changes as the input changes. For position over time, that rate is the velocity.

Watch out: the derivative is the slope of the **tangent** (at a single point), not the secant (between two points). The secant gives the **average** rate of change; the derivative gives the rate at one instant.

Practice

Try these in order. The methods are all above.

2.1 State what the derivative of a function tells you about its graph. [2]

2.2 State the difference between a secant line and a tangent line. [2]

2.3 State the two common notations for the derivative. [2]

2.4 A line has gradient 3. State the derivative of that line at every point. [1]

2.5 For an object's position over time, state what its derivative represents. [1]

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6 \arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25 + t^2}$.

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A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

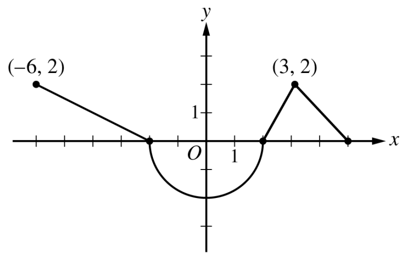
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is f increasing? Justify your answer.

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

2 The power rule – Part 2

Name: ____________ Class: ________ Date: __________

Recall

In Part A you learned that the derivative is the slope of a curve. Now compute it quickly with a rule:

- Finding a slope from scratch each time would be slow.
- A simple rule handles powers of x.

Each idea has a worked example before the Practice.

New ideas

1 The power rule

To differentiate a power of x, multiply by the power and lower the power by one:

$$\frac{d}{dx}x^n = n\,x^{n-1}.$$

Worked example. $\frac{d}{dx}x^3 = 3x^2$, and $\frac{d}{dx}x^5 = 5x^4$.

2 Constants

The derivative of a **constant** is 0 (a flat line has zero slope). A constant multiplier stays: $\frac{d}{dx}(5x^2) = 10x$.

3 Sums

Differentiate a sum term by term.

Worked example. $\frac{d}{dx}(x^2 + 3x) = 2x + 3$.

4 Reading the result

The derivative is itself a function – put in an x value to get the slope there.

Worked example. For $f(x) = x^2$, $f'(x) = 2x$, so the slope at $x = 3$ is $f'(3) = 6$.

Watch out: the power rule **lowers** the power by one and **multiplies** by the old power. A common slip is to forget the multiply – $\frac{d}{dx}x^4$ is $4x^3$, not x^3.

Practice

Try these in order. The methods are all above.

2.1 Find $\frac{d}{dx}x^4$. [2]

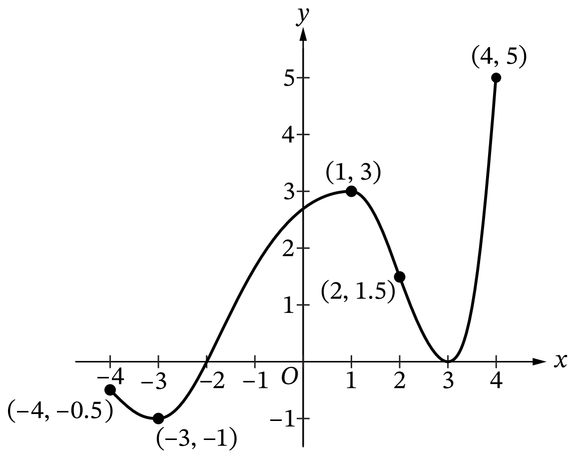
2.2 Find $\frac{d}{dx}x^7$. [1]

2.3 Find the derivative of $f(x) = x^2 + 5x$. [2]

2.4 Find the derivative of $f(x) = 3x^2$. [2]

2.5 For $f(x) = x^2$, find the slope of the curve at $x = 4$. [2]

4. Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f', the derivative of f, is shown.



Graph of f'

Part A

For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

Part B

Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

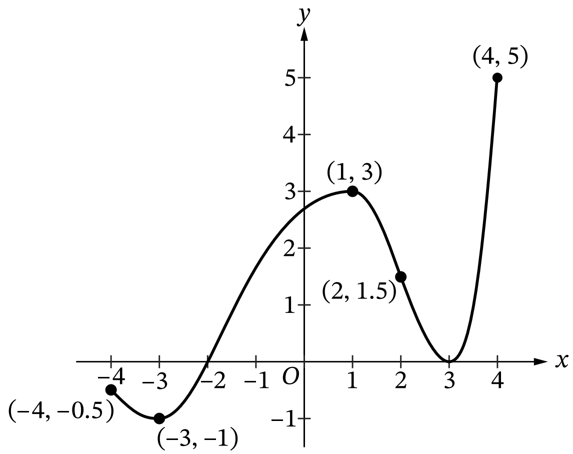
Part C

For $-4 \le x \le 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.

Part D

For $-4 \le x \le 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

4. Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f', the derivative of f, is shown.



Graph of f'

Part A

For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

Part B

Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Part C

For $-4 \le x \le 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.

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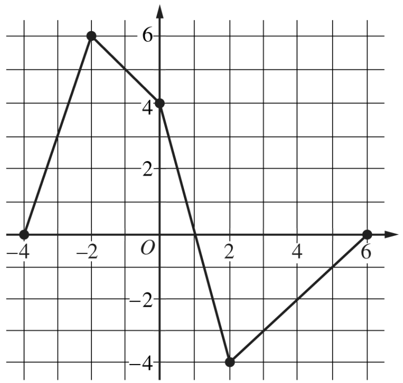
For $-4 \le x \le 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Let f be the function defined by $f(x) = \dfrac{\quad}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\sum\limits_{n=5}^{\infty} \dfrac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

3 The chain rule – Part 1

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
chain rule	链式法则	____________ ____________ ____________

Recall

In the last topic you used the power rule to differentiate powers of x:

- $\dfrac{d}{dx}x^3 = 3x^2$.
- But what about something like $(x^2+1)^3$ – a function **inside** another function?

This sheet gives the rule for that: the chain rule. Each idea has a worked example.

New ideas

1 Composite functions

A **composite function** is one function tucked **inside** another, like $(x^2+1)^3$: the "inside" is x^2+1 and the "outside" is "cube it".

2 The chain rule

To differentiate a composite function, use the **chain rule** 链式法则:

> differentiate the **outside** (keeping the inside as it is), then **multiply** by the derivative of the inside.

Worked example. For $(x^2+1)^3$: the outside gives $3(x^2+1)^2$, and the inside x^2+1 has derivative $2x$, so

$$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2\cdot 2x.$$

3 Another example

Worked example. For $(3x+1)^4$: outside $\to 4(3x+1)^3$; inside $3x+1$ has derivative 3; so the answer is $4(3x+1)^3\cdot 3 = 12(3x+1)^3$.

4 Why "chain"?

The rule links the rates together in a chain: how fast the outside changes with the inside, times how fast the inside changes with x.

Watch out: don't forget to multiply by the derivative of the **inside**. Writing just $3(x^2+1)^2$ (and forgetting the $\times 2x$) is the most common chain-rule mistake.

Practice

Try these in order. The methods are all above.

2.1 State, in words, what the chain rule tells you to do. [2]

2.2 For $(x^2+1)^3$, state the "inside" function and its derivative. [2]

2.3 Differentiate $(2x+5)^3$. [3]

2.4 Differentiate $(x^2+4)^2$. [3]

2.5 State the most common mistake when using the chain rule. [1]

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

   (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t)\,dt$.

   (b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

   (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

   Show the setup for your calculations.

   (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

3 The second derivative and inverse functions – Part 2

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
second derivative	二阶导数	

Recall

In Part A you met the chain rule. Now two more ideas:

- You can differentiate a derivative again.
- Some functions are the "reverse" of others (inverses).

Each idea has a worked example before the Practice.

New ideas

1 The second derivative

If you differentiate a function and then differentiate **again**, you get the **second derivative** 二阶导数, written $f''(x)$.

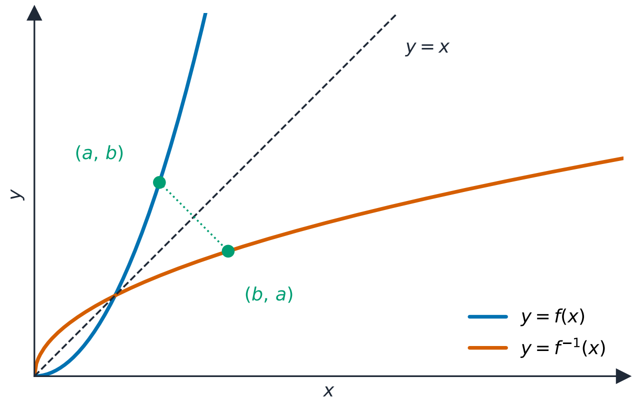
Worked example. For $f(x) = x^3$: the first derivative is $f'(x) = 3x^2$, and differentiating again gives $f''(x) = 6x$.

2 What it means

The first derivative is the **rate of change** (like velocity); the second derivative is the **rate of change of that rate** (like acceleration). It tells you whether the slope is increasing or decreasing.

3 Inverse functions

An **inverse function** reverses a function. Its graph is the reflection of the original in the line $y = x$.



4 Building up

Worked example. For $f(x) = x^2 + 3x$: $f'(x) = 2x + 3$, and $f''(x) = 2$ (a constant, so the slope changes at a steady rate).

Watch out: the second derivative is found by differentiating **twice** – first get $f'(x)$, then differentiate that. It is the derivative of the derivative, not the square of the derivative.

Practice

Try these in order. The methods are all above.

2.1 State what the second derivative is. [1]

2.2 For $f(x) = x^4$, find $f'(x)$ and then $f''(x)$. [3]

2.3 For position over time, state what the first and second derivatives represent. [2]

2.4 State how the graph of an inverse function relates to the original. [2]

2.5 For $f(x) = x^2$, find $f''(x)$. [2]

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for f about $x = 1$.

C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

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D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

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4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x + 1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

4 Motion - velocity and acceleration – Part 1

Name: ______________ Class: __________ Date: ____________

Vocabulary 词汇

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English	中文	Write it 3 times (English)		
velocity	速度			
acceleration	加速度			

Recall

The derivative is a rate of change – and motion is all about rates:

- Velocity is how fast your position changes.
- Acceleration is how fast your velocity changes.
- Calculus connects all three.

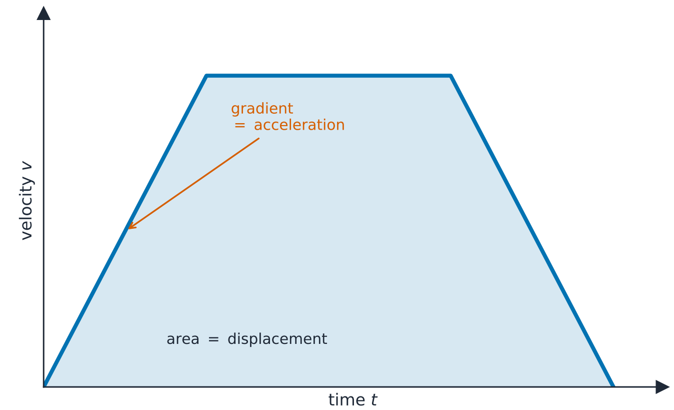
This sheet lines up motion with derivatives. Each idea has a worked example.

New ideas

■ 1 Position, velocity, acceleration

For an object moving along a line:

- **velocity** 速度 is the derivative of position (how fast position changes),
- **acceleration** 加速度 is the derivative of velocity (how fast velocity changes).



■ 2 From position to velocity

Worked example. If position is $s(t) = t^2$, then velocity is $s'(t) = 2t$. At $t = 3$ the velocity is $2 \times 3 = 6$.

■ 3 From velocity to acceleration

Worked example. With velocity $v(t) = 2t$, the acceleration is $v'(t) = 2$ – a constant.

■ 4 Direction from the sign

A **positive** velocity means moving in the positive direction; a **negative** velocity means moving the other way. The object is momentarily at rest when velocity is 0.

Watch out: velocity is the derivative of **position**, and acceleration is the derivative of **velocity** – so acceleration is the derivative of position **twice** (the second derivative).

Practice

Try these in order. The methods are all above.

2.1 State how velocity is related to position, and how acceleration is related to velocity. [2]

2.2 Position is $s(t) = t^2$. Find the velocity $s'(t)$. [2]

2.3 Using that velocity, find the velocity at $t = 5$. [2]

2.4 For velocity $v(t) = 3t$, find the acceleration. [2]

2.5 State what a velocity of 0 tells you about the object's motion. [1]

2. Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2e^{-0.0025h^2}$ for $0 \le h \le 30$ and is modeled by $f(h)$ for $h \ge 30$. The continuous function f is not explicitly given.

   (a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.

   (b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?

   (c) There is a function u such that $0 \le f(h) \le u(h)$ for all $h \ge 30$ and $\int_{30}^{\infty} u(h)\,dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.

   (d) The boat is moving on the surface of the sea. At time $t \ge 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \le t \le 1$.

END OF PART A OF SECTION II

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1 + x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

4 Related rates and linear approximation – Part 2

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
Related rates	相关变化率	______ ______ ______

Recall

In Part A you used derivatives for motion. Two more uses:

- When one thing changes, a connected thing changes too.
- A tangent line can estimate values near a known point.

Each idea has a worked example before the Practice.

New ideas

1 Related rates

Related rates 相关变化率 problems appear when two quantities change together over time. If you know how fast one changes, you can find how fast the other does.

Worked example. A square's side grows at 2 cm/s. Since area $A = s^2$, the area's rate of change is $\frac{dA}{dt} = 2s \cdot \frac{ds}{dt}$. When $s = 5$, $\frac{dA}{dt} = 2(5)(2) = 20$ cm^2/s.

2 Linear approximation

The tangent line at a point stays close to the curve nearby, so it gives a quick **estimate** of the function's value.

3 Using the tangent

Near a known point, $f(x) \approx f(a) + f'(a)(x - a)$ – follow the tangent line a small step.

Worked example. For $f(x) = x^2$ near $x = 3$: $f(3) = 9$ and $f'(3) = 6$, so $f(3.1) \approx 9 + 6(0.1) = 9.6$ (the true value is 9.61).

4 Why it works

Because a smooth curve looks almost straight when you zoom in, the tangent is an excellent approximation close to the point.

Watch out: a tangent-line estimate is only good **close** to the point of contact. Far from it, the curve bends away from the tangent and the approximation gets worse.

Practice

Try these in order. The methods are all above.

2.1 State what a related-rates problem involves. [2]

2.2 A square's side grows at 3 cm/s. Using $\frac{dA}{dt} = 2s\frac{ds}{dt}$, find the rate the area grows when $s = 4$. [3]

2.3 State what a linear approximation uses to estimate a function's value. [1]

2.4 For $f(x) = x^2$ near $x = 2$ ($f(2) = 4$, $f'(2) = 4$), estimate $f(2.1)$. [3]

2.5 State when a tangent-line approximation stops being accurate. [1]

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

5. Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$.

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Find $\int f(x)\,dx$.

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x + 1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

5 Increasing, decreasing, and turning points – Part 1

Name: ____________ Class: ________ Date: __________

Recall

The derivative gives the slope of a curve – and the slope tells you a lot:

- Where a graph goes uphill, the slope is positive.
- At the very top of a hill, the graph is momentarily flat.
- Businesses want to find the "best" (maximum or minimum) point.

This sheet uses the derivative to find high and low points. Each idea has a worked example.

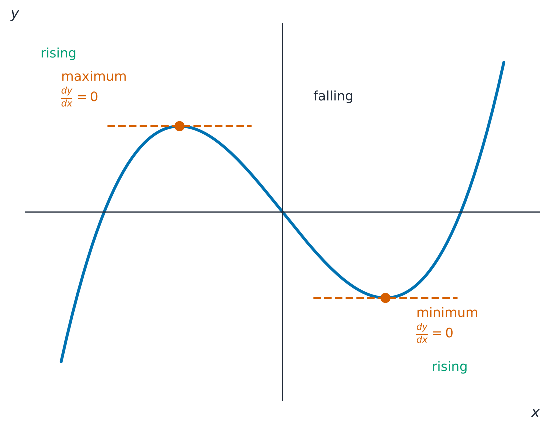
New ideas

1 Increasing and decreasing

- Where the derivative $f'(x) > 0$, the function is **increasing** (going uphill).
- Where $f'(x) < 0$, it is **decreasing** (going downhill).

2 Stationary points

Where $f'(x) = 0$, the curve is momentarily **flat** – a **stationary point** (a peak or a valley).



3 Finding a maximum or minimum

To find high and low points: differentiate, set $f'(x) = 0$, and solve for x.

Worked example. For $f(x) = x^2 - 4x$: $f'(x) = 2x - 4$. Setting $2x - 4 = 0$ gives $x = 2$ – the minimum point of this parabola.

4 Peak or valley?

Check the slope just before and after: if it changes from positive to negative, it is a **maximum**; from negative to positive, a **minimum**.

Watch out: a stationary point ($f' = 0$) can be a maximum **or** a minimum (or neither). Finding where $f'(x) = 0$ locates the point; you still have to check which kind it is.

Practice

Try these in order. The methods are all above.

2.1 State what the sign of $f'(x)$ tells you about the graph. [2]

2.2 State what is true about the derivative at a stationary point. [1]

2.3 For $f(x) = x^2 - 6x$, find $f'(x)$ and the x-value where it is zero. [3]

2.4 For $f(x) = x^2 + 2x$, find the x-value of the stationary point. [2]

2.5 State how you decide whether a stationary point is a maximum or a minimum. [2]

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

AP Calculus BC: **2015**

5. Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Find $\int f(x)\,dx$.

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1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

(a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

(d) A line forms whenever $A(t) \ge 400$. The number of vehicles in line at time t, for $a \le t \le 4$, is given by $N(t) = \int_a^t (A(x) - 400)\,dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \le t \le 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Let f be the function defined by $f(x) = \frac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\frac{3}{2x^2 - 7x + 5} = \frac{2}{2x - 5} - \frac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x + 1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

5 Concavity and the shape of a curve – Part 2

Name: ____________ Class: ________ Date: __________

Recall

In Part A you found peaks and valleys with the first derivative. Now the **second** derivative describes the curve's bend:

- A curve can bend upward (like a smile) or downward (like a frown).
- The second derivative tells you which.

Each idea has a worked example before the Practice.

New ideas

1 Concavity

The way a curve bends is its **concavity**:

- $f''(x) > 0$: **concave up** (a smile / valley shape),
- $f''(x) < 0$: **concave down** (a frown / hill shape).

2 Points of inflection

Where the curve **changes** its bend (from concave up to concave down or the reverse) is a **point of inflection** – there $f''(x) = 0$.

3 The second-derivative test

At a stationary point ($f' = 0$): if $f'' > 0$ it is a **minimum** (valley), and if $f'' < 0$ it is a **maximum** (hill).

Worked example. For $f(x) = x^2 - 4x$: $f'(x) = 2x - 4 = 0$ at $x = 2$, and $f''(x) = 2 > 0$, so $x = 2$ is a **minimum**.

4 Putting it together

The first derivative finds where the peaks and valleys are; the second derivative tells you which is which and how the curve bends.

Watch out: concave up is a "smile" (holds water) and concave down is a "frown" (spills water). A concave-up curve can still be going downhill – concavity is about the **bend**, not the slope.

Practice

Try these in order. The methods are all above.

2.1 State what $f''(x) > 0$ and $f''(x) < 0$ tell you about a curve's shape. [2]

2.2 State what a point of inflection is. [2]

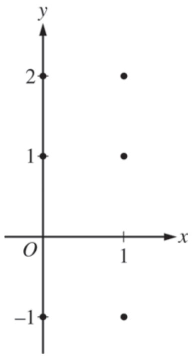
2.3 For $f(x) = x^2 + 6x$: find $f'(x)$, the stationary point, and use $f''(x)$ to say if it is a max or min. [3]

2.4 State the second-derivative test for a minimum. [1]

2.5 Explain why "concave up" does not always mean the curve is going uphill. [2]

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

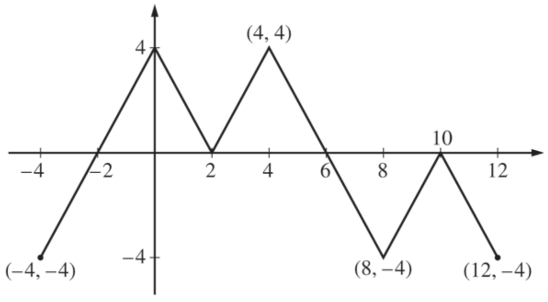
(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x + 1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

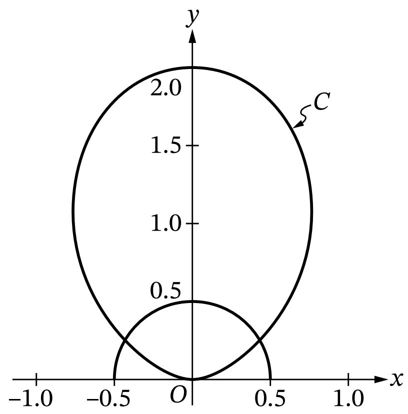
(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

2. Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.



(Note: Your calculator should be in radian mode.)

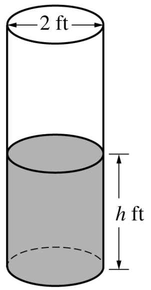
A. Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.

B. Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.

C. It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

D. A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

END OF PART A



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

6 The area under a curve – Part 1

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
Riemann sum	黎曼和	

Recall

You can find the area of simple shapes:

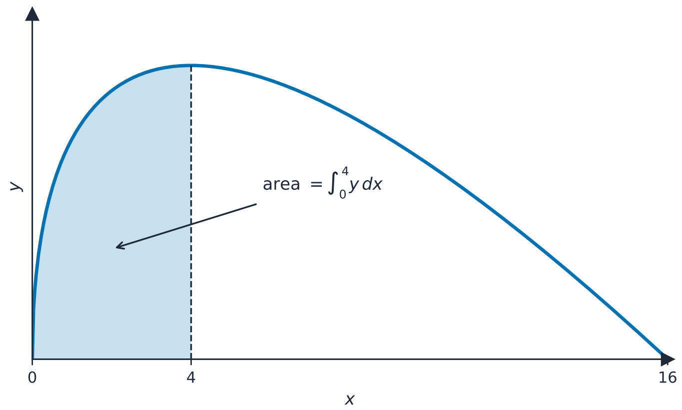
- The area of a rectangle is width $\times$ height.
- But how do you find the area under a **curve**, which has no straight edges?

This sheet introduces the idea behind **integration**. Each idea has a worked example.

New ideas

■ 1 Area under a curve

The **area under a curve** (between the curve and the x-axis) is a useful quantity – for example, the area under a velocity–time graph is the distance travelled.



■ 2 Approximating with rectangles

You can estimate that area by covering it with thin **rectangles** and adding their areas – a **Riemann sum** 黎曼和.

■ 3 More rectangles, better estimate

The thinner the rectangles (the more you use), the closer the total gets to the true area.

Worked example. To estimate the area under $y = x$ from 0 to 4 with rectangles: the exact area is a triangle, $\frac{1}{2} \times 4 \times 4 = 8$, and more rectangles get closer and closer to this.

■ 4 The exact area

Making the rectangles infinitely thin gives the **exact** area – this exact value is the **integral**.

Watch out: a Riemann sum with a few rectangles is only an **estimate**. The exact area (the integral) is the limit as the rectangles become infinitely thin and infinitely many.

Practice

Try these in order. The methods are all above.

2.1 State what "the area under a curve" means. [2]

2.2 State how a Riemann sum estimates that area. [2]

2.3 State what happens to the estimate as you use more, thinner rectangles. [1]

2.4 Find the exact area under $y = x$ from 0 to 6 (it is a triangle). [3]

2.5 State what the area under a velocity–time graph represents. [1]

1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0,10]$, $[10,20]$, and $[20,30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t+15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

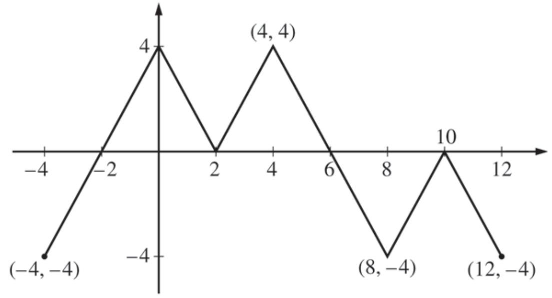
B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

6 Integration and the power rule – Part 2

Name: ______ Class: ______ Date: ______

Recall

In Part A you saw that the integral is the exact area under a curve. Now compute it:

- Integration is the **reverse** of differentiation.
- There is a simple rule for powers of x, just like the power rule for derivatives.

Each idea has a worked example before the Practice.

New ideas

1 Integration undoes differentiation

Integration finds a function whose derivative is the one you started with – it is the reverse of differentiating. This reverse is written with the sign $\int$.

2 The power rule for integration

To integrate a power of x, **raise the power by one and divide by the new power**:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Worked example. $\int x^2\,dx = \dfrac{x^3}{3} + C$ (check: differentiating $\frac{x^3}{3}$ gives x^2).

3 The "+ C"

Because many functions share the same derivative (they differ by a constant), we add a constant C when integrating.

4 Definite integral gives area

A **definite integral** (with limits) gives the exact area under the curve between two x-values.

Worked example. The area under $y = x$ from 0 to 2 is $\left[\frac{x^2}{2}\right]_0^2 = \frac{4}{2} - 0 = 2$.

Watch out: integration **raises** the power and **divides** – the exact opposite of differentiation, which lowers the power and multiplies. Mixing them up is a common slip.

Practice

Try these in order. The methods are all above.

2.1 State how integration relates to differentiation. [1]

2.2 Find $\int x^3\,dx$. [2]

2.3 Find $\int x^4\,dx$. [2]

2.4 State why we add "+C" when integrating. [2]

2.5 Find the area under $y = x$ from 0 to 4 using $\left[\frac{x^2}{2}\right]_0^4$. [3]

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

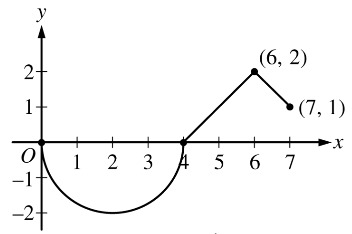
(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f'

3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

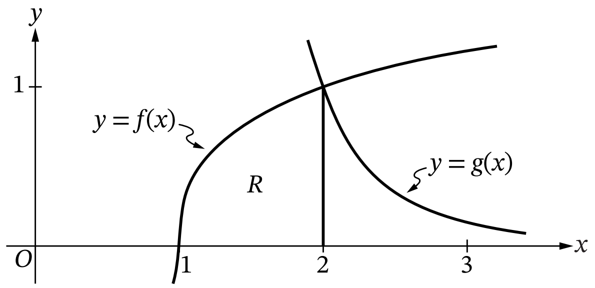
(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Let f be the function defined by $f(x) = \sqrt[3]{x-1}$, and let g be the function defined by $g(x) = e^{(-2x+4)}$. The graphs of f and g intersect at the point $(2, 1)$. Let R be the region bounded by the graph of f, the x-axis, and the vertical line $x = 2$, as shown in the figure.



Part A

Evaluate $\int_1^2 f(x)\,dx$. Show the work that leads to your answer.

Part B

Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region R is rotated about the x-axis.

Part C

Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region R.

Part D

Evaluate $\int_2^\infty g(x)\,dx$. Show the work that leads to your answer.

x	0	π	2π
$f'(x)$	5	6	0

5. The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

(a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\,dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

(b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\,dx$ provide about the graph of f ?

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t+5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^{\infty} f(x)\, dx$ or show that the integral diverges.

(d) Determine whether the series $\sum_{n=5}^{\infty} \dfrac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

AP Calculus BC: **2015**

5. Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$.

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Find $\int f(x)\, dx$.

Name: AP Calculus BC: **2017**

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^{\infty} f(x)\, dx$ or show that the integral diverges.

(d) Determine whether the series $\sum_{n=5}^{\infty} \dfrac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

7 Differential equations and slope fields – Part 1

Name: ____________ Class: ________ Date: __________

Recall

You have solved ordinary equations to find a number:

- An equation like $2x + 1 = 7$ has the solution $x = 3$.
- But some equations describe how a quantity **changes** – they involve a derivative.

This sheet introduces differential equations. Each idea has a worked example.

New ideas

1 What a differential equation is

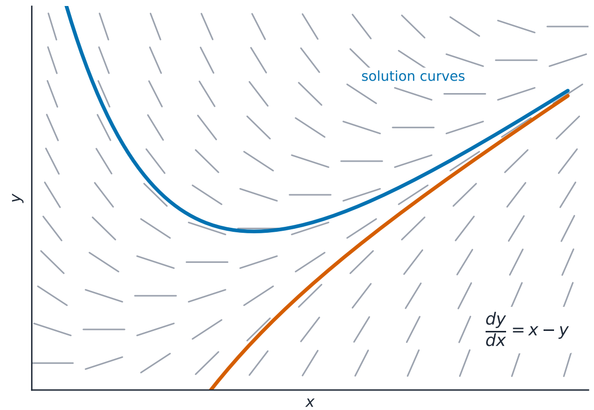
A **differential equation** involves a **derivative** – it tells you the **rate of change**, not the value directly. For example $\dfrac{dy}{dx} = 2x$ says "the slope at each point is $2x$".

2 Its solution is a function

Solving it means finding the **function** y whose derivative matches. For $\dfrac{dy}{dx} = 2x$, one solution is $y = x^2$ (its derivative is $2x$).

3 Slope fields

A **slope field** draws a tiny line at many points showing the slope there. It lets you see the shape of the solutions without solving.



4 A family of solutions

There are usually **many** solutions differing by a constant (like $y = x^2$, $y = x^2 + 1$, ...) – a whole family that follows the slope field.

Watch out: a differential equation gives the **slope** (dy/dx), not the function itself. You must integrate (or otherwise solve) to recover the actual function y.

Practice

Try these in order. The methods are all above.

2.1 State what makes an equation a "differential equation". [2]

2.2 State what it means to solve a differential equation. [2]

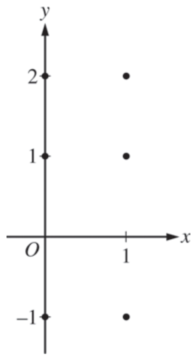
2.3 For $\frac{dy}{dx} = 2x$, verify that $y = x^2$ is a solution. [2]

2.4 State what a slope field shows. [2]

2.5 State why a differential equation usually has many solutions. [1]

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



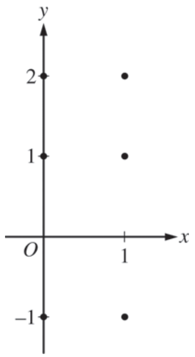
(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



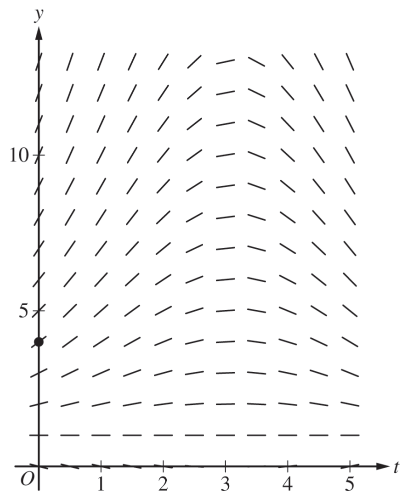
(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

7 Solving simple differential equations – Part 2

Name: ______________ Class: __________ Date: ____________

Recall

In Part A you met differential equations. Now solve simple ones:

- Solving means finding the function from its rate of change.
- One special kind describes things that grow by a fixed percentage – like money or populations.

Each idea has a worked example before the Practice.

New ideas

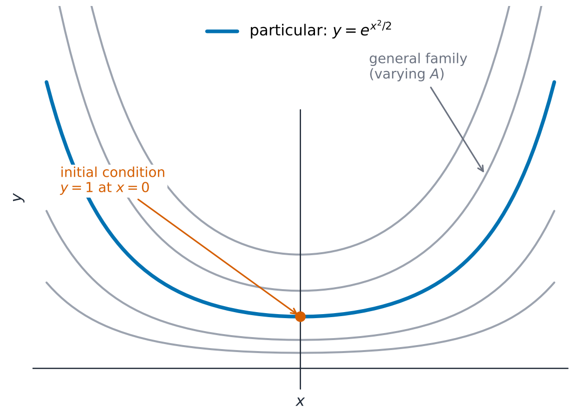
1 Solving by integrating

If $\frac{dy}{dx}$ is given, integrate to find y.

Worked example. $\frac{dy}{dx} = 3x^2$ integrates to $y = x^3 + C$.

2 A family of curves

The constant C gives a **family** of solution curves; one extra fact (a known point) picks out the right one.



3 Exponential growth

A very common equation is $\frac{dy}{dt} = ky$ – "the rate of growth is proportional to how much there is". Its solution is exponential growth, $y = Ce^{kt}$.

4 Where it appears

This models populations, money earning interest, and radioactive decay (with a negative k) – anything that grows or shrinks by a fixed proportion.

Watch out: exponential growth ($\frac{dy}{dt} = ky$) means the quantity grows **faster and faster** as it gets bigger, because the rate depends on the current amount. It is not steady, straight-line growth.

Practice

Try these in order. The methods are all above.

2.1 State how you solve $\frac{dy}{dx} = f(x)$ for y. [1]

2.2 Solve $\dfrac{dy}{dx} = 2x$. [2]

2.3 Solve $\dfrac{dy}{dx} = 4x^3$. [2]

2.4 State the type of growth described by $\dfrac{dy}{dt} = ky$. [1]

2.5 Give two real situations modelled by exponential growth or decay. [2]

5. Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = (3-x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for f about $x = 1$.

C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

AP Calculus BC: **2016**

4. Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(a) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim\limits_{x \to -1}\left(\dfrac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

8 Area between two curves – Part 1

Name: ____________ Class: ________ Date: __________

Recall

You have found the area under a single curve. Now the area **between** two curves:

- Two curves can enclose a region between them.
- Finding that area is a common use of integration.

Each idea has a worked example before the Practice.

New ideas

1 Area between two curves

The area between an upper curve and a lower curve is the integral of the **difference**:

$$\text{area} = \int (\text{top} - \text{bottom})\, dx.$$

2 Why subtract

The area under the top curve minus the area under the bottom curve leaves exactly the region between them.

3 A worked example

Worked example. Between $y = 4$ (top) and $y = x^2$ (bottom), where they meet at $x = -2$ and $x = 2$, the area is $\int_{-2}^{2}(4 - x^2)\, dx$ – the constant 4 line minus the parabola.

4 Getting the limits

The two x-values where the curves **cross** are the limits of the integral – set the two functions equal to find them.

Watch out: always subtract **top minus bottom** (the higher curve minus the lower). Subtracting the wrong way round gives a **negative** area, which is a sign you have them reversed.

Practice

Try these in order. The methods are all above.

2.1 Write the formula for the area between an upper and a lower curve. [2]

2.2 Explain why you subtract the two curves. [2]

2.3 For the region between $y = 6$ and $y = x^2$, state what you would integrate. [2]

2.4 State how you find the limits of the integral. [2]

2.5 State what a negative area tells you about your set-up. [1]

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25+t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\,dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\,dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

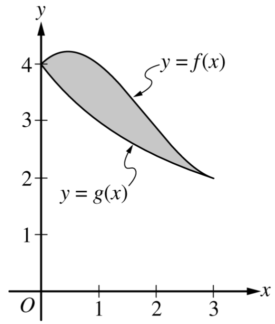
t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.



5. The graphs of the functions f and g are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \frac{12}{3+x}$ for $x \ge 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

8 Volumes and average value – Part 2

Name: ____________ Class: ________ Date: __________

Recall

In Part A you found areas with integration. Integration does even more:

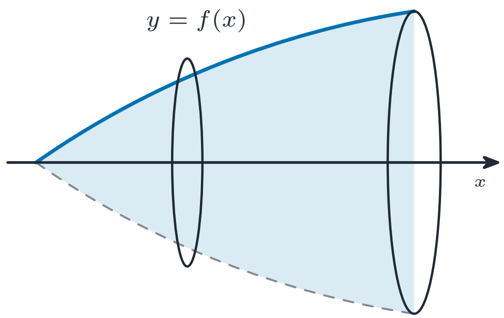
- Spin a flat region around a line and it sweeps out a 3-D solid.
- Integration can find that solid's volume, and the average value of a function.

Each idea has a worked example before the Practice.

New ideas

1 Volumes of revolution

If you spin a region around an axis, it sweeps out a **solid of revolution** – like spinning a rectangle to make a cylinder, or a curve to make a vase shape.



rotate the region around the x-axis to sweep a solid

2 Slicing into discs

Integration finds the volume by adding up many thin **discs**, each a circle of area πr^2 times a tiny thickness.

3 Average value

The **average value** of a function over an interval is its integral divided by the width of the interval – the "typical" height of the curve.

Worked example. The average value of $y = x$ from 0 to 4 is $\frac{1}{4}\int_0^4 x\,dx = \frac{1}{4} \times 8 = 2$.

4 Integration as "adding up"

The common thread: integration **adds up infinitely many tiny pieces** – tiny areas, tiny discs, or tiny contributions to an average.

Watch out: integration is best understood as "adding up many tiny pieces" – tiny rectangles for area, tiny discs for volume. Whenever a quantity is a sum of infinitely many small parts, an integral finds it.

Practice

Try these in order. The methods are all above.

2.1 State what a solid of revolution is. [2]

2.2 State the shape you get by spinning a rectangle around one of its sides. [1]

2.3 State how integration finds the volume of a solid of revolution. [2]

2.4 Find the average value of $y = x$ from 0 to 6 (using $\frac{1}{6}\int_0^6 x\,dx$ with $\int_0^6 x\,dx = 18$). [3]

2.5 In one sentence, state what integration does in all these cases. [1]

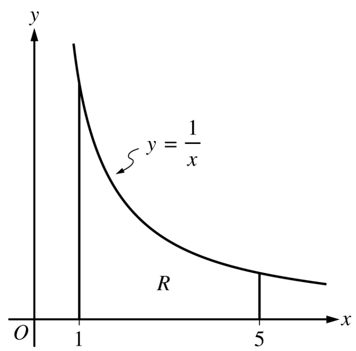


Figure 1

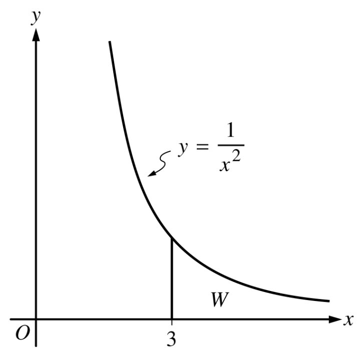


Figure 2

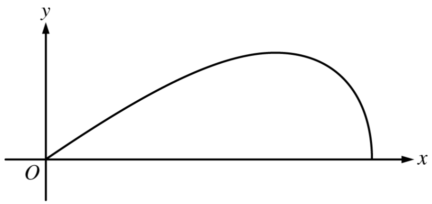
5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

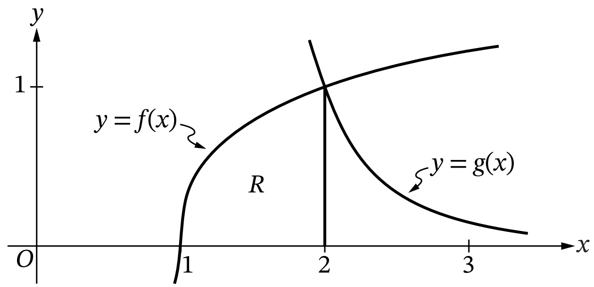
(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Let f be the function defined by $f(x) = \sqrt[3]{x - 1}$, and let g be the function defined by $g(x) = e^{(-2x + 4)}$. The graphs of f and g intersect at the point $(2, 1)$. Let R be the region bounded by the graph of f, the x-axis, and the vertical line $x = 2$, as shown in the figure.



Part A

Evaluate $\int_1^2 f(x)dx$. Show the work that leads to your answer.

Part B

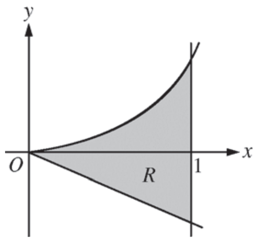
Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region R is rotated about the x-axis.

Part C

Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region R.

Part D

Evaluate $\int_2^\infty g(x)dx$. Show the work that leads to your answer.



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

x	0	π	2π
$f'(x)$	5	6	0

5. The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

(a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

(b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t + 5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

9 Parametric and vector functions – Part 1

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
parametric function	参数方程	
vector	向量	

Recall

So far a function gave one output for each input, like $y = f(x)$:

- But a moving object needs both its across (x) and up (y) positions.
- Both can be given in terms of time.

This sheet introduces parametric and vector functions. Each idea has a worked example.

New ideas

1 Parametric functions

A **parametric function** 参数方程 gives both coordinates in terms of a third variable (a **parameter**), usually time t: $\big(x(t),\, y(t)\big)$. As t runs, the point traces a path.

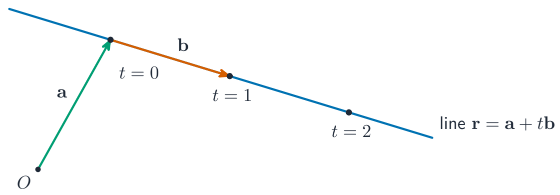
Worked example. $x(t) = t$, $y(t) = t^2$: at $t = 0$ the point is $(0, 0)$; at $t = 1$, $(1, 1)$; at $t = 2$, $(2, 4)$ – tracing a parabola.

2 Direction of travel

Because t is like time, the path has a **direction** – the order in which it is drawn as t increases.

3 Vectors

A **vector** 向量 has magnitude and direction, written by components $\langle a, b\rangle$. A **vector-valued function** packages a moving point as $\vec{r}(t) = \langle x(t), y(t)\rangle$.



4 Same idea, two forms

A parametric function and a vector-valued function carry the **same** information – the position of a moving point at each time t.

Watch out: in a parametric curve, y is **not** given directly in terms of x – both are given in terms of t. To find the ordinary y-vs-x shape you eliminate t.

Practice

Try these in order. The methods are all above.

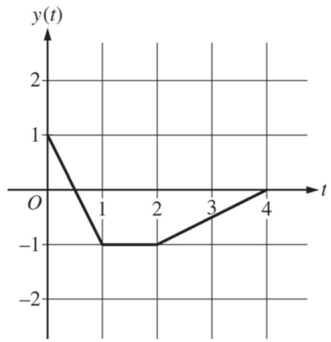
2.1 State what a parametric function gives you. [2]

2.2 For $x(t) = t$, $y(t) = t^2$, find the point when $t = 3$. [2]

2.3 For $x(t) = 2t$, $y(t) = t + 1$, find the point when $t = 0$ and when $t = 2$. [2]

2.4 State the two things a vector has. [2]

2.5 State what a parametric and a vector-valued function have in common. [1]



2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

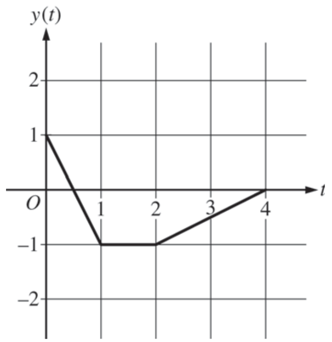
(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

END OF PART A OF SECTION II



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(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

END OF PART A OF SECTION II

2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

(c) Find the time at which the speed of the particle is 3.

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

END OF PART A OF SECTION II

2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

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   (b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

   (c) Find the time at which the speed of the particle is 3.

   (d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

END OF PART A OF SECTION II

AP CALCULUS BC • HOLIDAY HOMEWORK 假期作业

9 Polar coordinates – Part 2

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	**中文**	**Write it 3 times** (English)
Polar coordinates	极坐标	________ ________ ________

Recall

You locate points on a grid with (x, y) coordinates:

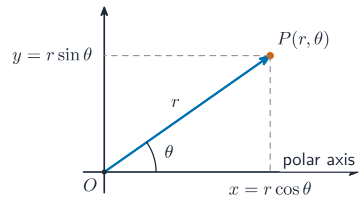
- x is how far across, y is how far up.
- But you could also describe a point by its **distance** and **direction** from the centre.

Each idea has a worked example before the Practice.

New ideas

■ 1 Polar coordinates

Polar coordinates 极坐标 locate a point by its distance r from the origin and its angle θ from the positive x-axis – written (r, θ).



2 Converting to x and y

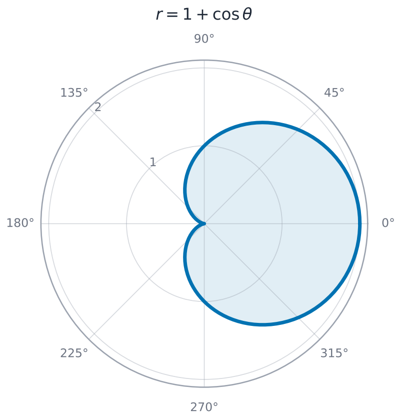
To convert polar to the usual coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Worked example. The polar point $(2, 0°)$ is $x = 2\cos 0° = 2$, $y = 2\sin 0° = 0$ – the point $(2, 0)$.

3 Polar curves

Letting r depend on θ traces a **polar curve** – often beautiful shapes like circles, spirals, and flower petals.



4 Why use polar

Polar coordinates make round and rotating shapes much simpler to describe than x–y coordinates do.

Watch out: in polar coordinates the first number is the **distance** r and the second is the **angle** θ – not an x and a y. Reading (2, 90°) as "across 2, up 90" is wrong; it means "distance 2, angle 90°".

Practice

Try these in order. The methods are all above.

2.1 State how polar coordinates locate a point. [2]

2.2 Write the formulas that convert polar (r, θ) to x and y. [2]

2.3 Convert the polar point $(3, 0°)$ to x–y coordinates. [2]

2.4 Convert the polar point $(4, 90°)$ to x–y coordinates ($\cos 90° = 0$, $\sin 90° = 1$). [3]

2.5 State one advantage of polar coordinates. [1]

2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

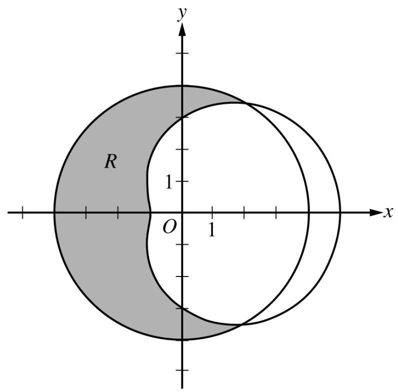
   (a) Find the x-coordinate of the position of the particle at time $t = 2$.

   (b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

   (c) Find the time at which the speed of the particle is 3.

   (d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

END OF PART A OF SECTION II

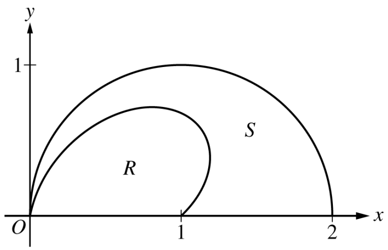


5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

   (a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

   (b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

   (c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

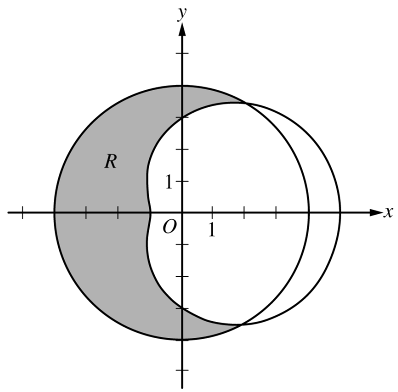
   (a) Find the area of R.

   (b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

   (c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

   (d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

END OF PART A OF SECTION II



5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

   (a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

   (b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

   (c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

AP CALCULUS BC • HOLIDAY HOMEWORK 假期作业

10 Infinite series – Part 1

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
sequence	数列	
series	级数	
geometric series	等比级数	

Recall

You met geometric sequences earlier – each term multiplies by a fixed ratio:

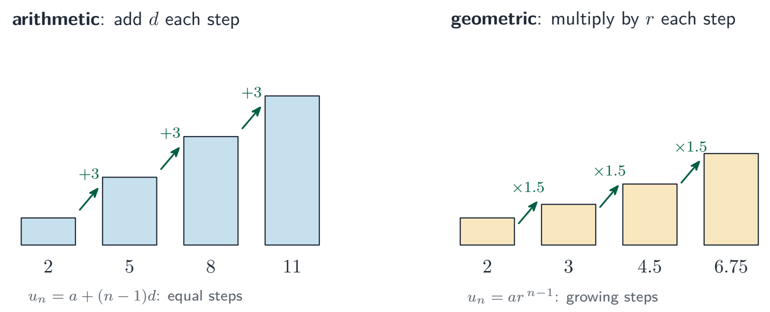
- $2, 4, 8, 16, \ldots$ doubles each time.
- What if you add up **all** the terms? Can an infinite sum have a finite answer?

This sheet introduces infinite series. Each idea has a worked example.

New ideas

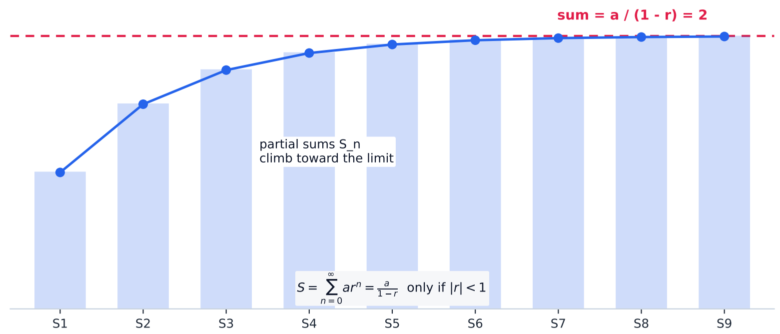
1 Sequence versus series

A **sequence** 数列 is a **list** of terms; a **series** 级数 is the **sum** of those terms.



2 Geometric series

A **geometric series** 等比级数 adds terms that each multiply by a ratio r: $a + ar + ar^2 + \cdots$.



3 Infinite sum

If $|r| < 1$, the terms shrink toward zero and the infinite sum **converges** to a finite value:

$$S = \frac{a}{1-r}.$$

Worked example. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ has $a = 1$, $r = \frac{1}{2}$, so $S = \dfrac{1}{1 - \frac{1}{2}} = 2$.

4 Converge or diverge

If $|r| \geq 1$ the terms do not shrink, so the sum grows without limit – it **diverges** (no finite answer).

Watch out: an infinite sum only has a finite value when the terms **shrink fast enough** ($|r| < 1$ for a geometric series). If the terms stay the same size or grow, the sum diverges.

Practice

Try these in order. The methods are all above.

2.1 State the difference between a sequence and a series. [2]

2.2 Write the formula for the sum of an infinite geometric series with $|r| < 1$. [1]

2.3 Find the sum of $1 + \frac{1}{3} + \frac{1}{9} + \cdots$ (with $a = 1$, $r = \frac{1}{3}$). [3]

2.4 Find the sum of $4 + 2 + 1 + \frac{1}{2} + \cdots$ (with $a = 4$, $r = \frac{1}{2}$). [2]

2.5 State whether $1 + 2 + 4 + 8 + \cdots$ converges, and why. [2]

6. The Taylor series for a function f about $x = 4$ is given by $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

STOP

END OF EXAM

5. Let f be the function defined by $f(x) = \dfrac{\text{[illegible]}}{2x^2 - 7x + 5}$.

   (a) Find the slope of the line tangent to the graph of f at $x = 3$.

   (b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

   (c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^{\infty} f(x)\,dx$ or show that the integral diverges.

   (d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

   (a) State the conditions necessary to use the integral test to determine convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$. Use the integral test to show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

   (b) Use the limit comparison test with the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

   (c) Determine the radius of convergence of the Maclaurin series for g.

   (d) The first two terms of the series $g(1) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

STOP

END OF EXAM

6. The Taylor series for a function f about $x = 4$ is given by $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2\cdot 3} + \frac{(x-4)^3}{3\cdot 3^2} + \frac{(x-4)^4}{4\cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

STOP

END OF EXAM

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\, dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

STOP

END OF EXAM

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10 Power and Taylor series – Part 2

Name: ________ Class: ________ Date: ________

Recall

In Part A you added up infinitely many terms. Now use that to **approximate functions**:

- A complicated function can be replaced by a simple polynomial.
- Adding more terms makes the approximation better.

Each idea has a worked example before the Practice.

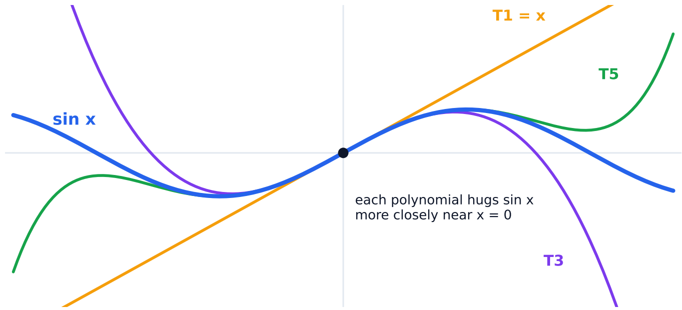
New ideas

1 Power series

A **power series** adds up powers of x: $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$. It behaves like an "infinitely long polynomial".

2 Taylor and Maclaurin series

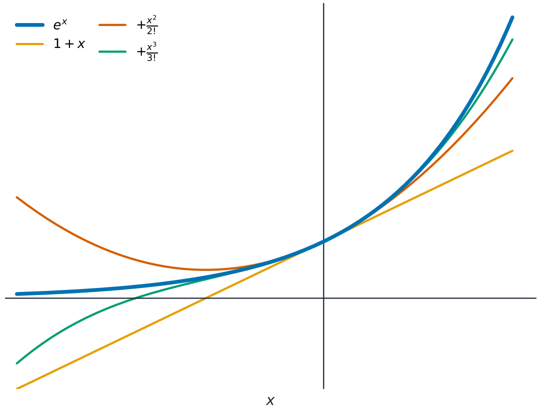
A **Taylor series** (a **Maclaurin series** when centred at 0) builds a power series that matches a function near a point – approximating it by a polynomial.



$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \quad \cdot \quad \text{for } \sin x\text{: } x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

■ 3 More terms, better fit

The more terms you keep, the more closely the polynomial hugs the real function.



■ 4 Why it matters

Polynomials are easy to add, differentiate, and integrate, so replacing a hard function with its series makes calculations far simpler – calculators use this idea.

Watch out: a Taylor approximation is most accurate **near the point** it is built around. Far from that point, you need more terms to keep it accurate.

Practice

Try these in order. The methods are all above.

2.1 State what a power series is. [2]

2.2 State what a Taylor (or Maclaurin) series does. [2]

2.3 State what happens to the approximation as you keep more terms. [1]

2.4 Explain why replacing a function with its power series is useful. [2]

2.5 State where a Taylor approximation is most accurate. [1]

6. The Maclaurin series for $\ln(1+x)$ is given by

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n+1}\frac{x^n}{n}+\cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x)=x\ln\left(1+\frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x=0$. Use the alternating series error bound to find an upper bound for $\left|P_4(2)-f(2)\right|$.

STOP

END OF EXAM

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5. Let $y=f(x)$ be the particular solution to the differential equation $\frac{dy}{dx}=(3-x)y^2$ with initial condition $f(1)=-1$.

A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1,-1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for f about $x=1$.

C. The second-degree Taylor polynomial for f about $x=1$ is used to approximate $f(1.1)$. Given that $|f'''(x)|\le 60$ for all x in the interval $1\le x\le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x=1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for f about $x = 1$.

C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \le 60$ for all x in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

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A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

STOP

END OF EXAM

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A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

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STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

STOP

END OF EXAM