HOLIDAY HOMEWORK 假期作业

AP Calculus BC

Exercise sheets, topic by topic.

每个单元：练习卷。

For class or self-study • no answer keys included.

Contents 目录

1 What a limit is – Part 1

Name: ____________ Class: ________ Date: ________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
limit	极限	____________ ____________ ____________

Recall

You have looked at graphs and how they behave:

- You can trace a curve and see where it is heading.
- Some graphs get closer and closer to a value without quite reaching it.
- You know how to put a number into a formula.

This sheet introduces the **limit** –the first big idea of calculus. Each idea has a worked example.

New ideas

1 What a limit is

A **limit** 极限 is the value a function **approaches** as x gets close to some point. We write it

$$\lim_{x \to a} f(x).$$

It asks: as x heads toward a, what does $f(x)$ head toward?

2 Finding a limit

If the function has no break at a, you can just substitute.

Worked example. $\lim_{x \to 2}(x^2 + 1) = 2^2 + 1 = 5$.

3 A hole in the graph

A limit can exist even if the function has a **hole** at that point –the graph still heads toward a value from both sides.

■ **4 From both sides**

The limit exists only if the function heads toward the **same** value from the left and from the right.

Watch out: the limit is about what $f(x)$ **approaches**, not necessarily the value at that exact point. A function can approach 5 even if, right at that point, it has a hole or a different value.

Practice

Try these in order. The methods are all above.

2.1 In words, state what $\lim_{x\to a} f(x)$ means. [2]

2.2 Find $\lim_{x\to 3}(x+4)$. [2]

2.3 Find $\lim_{x\to 2}(x^2-1)$. [2]

2.4 State whether a limit can exist at a point where the graph has a hole. [1]

2.5 State the condition on the left and right sides for a limit to exist. [2]

1 Continuity and asymptotes – Part 2

Name: ______ Class: ______ Date: ______

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)		
continuous	连续			
vertical asymptote	渐近线			

Recall

In Part A you met limits. Now use them to describe how smooth a graph is:

- Some graphs you can draw without lifting your pen.
- Others have jumps, holes, or shoot off to infinity.

Each idea has a worked example before the Practice.

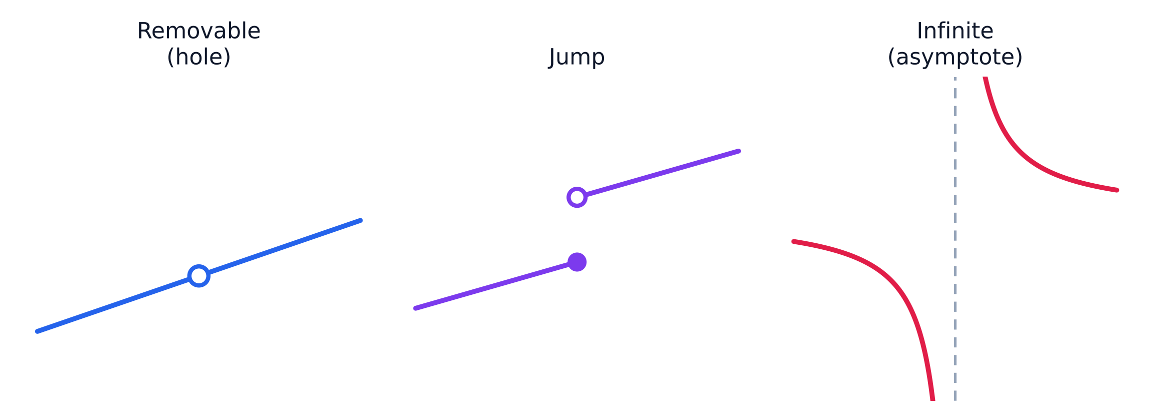
New ideas

1 Continuity

A function is **continuous** 连续 at a point if it has no break there –no hole, jump, or gap. A continuous graph can be drawn without lifting your pen.

2 Discontinuities

Where a graph breaks, it is **discontinuous**. The main kinds are a **hole**, a **jump**, and an **asymptote**.



■ 3 Vertical asymptotes

A **vertical asymptote** 渐近线 is a line the graph shoots up (or down) toward, without ever touching –it happens where the function blows up to infinity (like dividing by zero).

■ 4 Horizontal asymptotes

A **horizontal asymptote** is a value the graph levels off toward as x gets very large (or very negative).

Watch out: a graph can **cross** a horizontal asymptote but never crosses a **vertical** one –near a vertical asymptote the function races off to $\pm\infty$ instead.

Practice

Try these in order. The methods are all above.

2.1 State what it means for a function to be continuous at a point. [2]

2.2 Name three kinds of discontinuity. [3]

2.3 State what happens to a function near a vertical asymptote. [2]

2.4 State what a horizontal asymptote describes. [2]

2.5 For $f(x) = \dfrac{1}{x-3}$, state where the vertical asymptote is. [1]

2 The derivative as a slope – Part 1

Name: ______________ Class: __________ Date: ____________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)		
derivative	导数			
tangent	切线			

Recall

You already know about the gradient (slope) of a line:

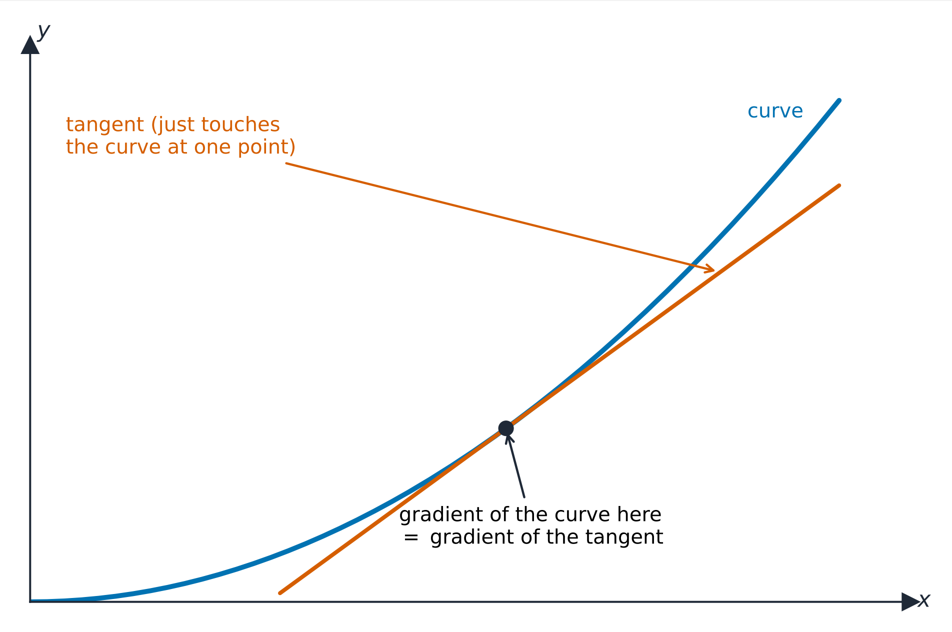
- A steep line has a big gradient; a flat one has zero.
- Gradient = rise over run.
- But a curve's steepness keeps changing along it.

This sheet introduces the **derivative** –the slope of a curve. Each idea has a worked example.

New ideas

■ 1 The slope of a curve

The **derivative** 导数 of a function at a point is the **slope of the curve** there –how steep it is at that exact spot.



■ 2 From secant to tangent

Take two points on the curve and join them –the **secant line**. Its slope is the **average** rate of change. As the two points move closer together, the secant becomes the **tangent** 切线 line, whose slope is the derivative.

■ 3 Notation

The derivative of f is written $f'(x)$ or $\dfrac{dy}{dx}$.

■ 4 A rate of change

The derivative is a **rate of change**: how fast the output changes as the input changes. For position over time, that rate is the velocity.

Watch out: the derivative is the slope of the **tangent** (at a single point), not the secant (between two points). The secant gives the **average** rate of change; the derivative gives the rate at one instant.

Practice

Try these in order. The methods are all above.

2.1 State what the derivative of a function tells you about its graph. [2]

2.2 State the difference between a secant line and a tangent line. [2]

2.3 State the two common notations for the derivative. [2]

2.4 A line has gradient 3. State the derivative of that line at every point. [1]

2.5 For an object's position over time, state what its derivative represents. [1]

2 The power rule – Part 2

Name: ______________ Class: __________ Date: ____________

Recall

In Part A you learned that the derivative is the slope of a curve. Now compute it quickly with a rule:

- Finding a slope from scratch each time would be slow.
- A simple rule handles powers of x.

Each idea has a worked example before the Practice.

New ideas

■ 1 The power rule

To differentiate a power of x, multiply by the power and lower the power by one:

$$\frac{d}{dx}\,x^n = n\,x^{n-1}.$$

Worked example. $\frac{d}{dx}\,x^3 = 3x^2$, and $\frac{d}{dx}\,x^5 = 5x^4$.

■ 2 Constants

The derivative of a **constant** is 0 (a flat line has zero slope). A constant multiplier stays: $\frac{d}{dx}(5x^2) = 10x$.

■ 3 Sums

Differentiate a sum term by term.

Worked example. $\frac{d}{dx}(x^2 + 3x) = 2x + 3$.

■ 4 Reading the result

The derivative is itself a function –put in an x value to get the slope there.

Worked example. For $f(x) = x^2$, $f'(x) = 2x$, so the slope at $x = 3$ is $f'(3) = 6$.

Watch out: the power rule **lowers** the power by one and **multiplies** by the old power. A common slip is to forget the multiply $-\frac{d}{dx}x^4$ is $4x^3$, not x^3.

Practice

Try these in order. The methods are all above.

2.1 Find $\dfrac{d}{dx}x^4$. [2]

2.2 Find $\dfrac{d}{dx}x^7$. [1]

2.3 Find the derivative of $f(x) = x^2 + 5x$. [2]

2.4 Find the derivative of $f(x) = 3x^2$. [2]

2.5 For $f(x) = x^2$, find the slope of the curve at $x = 4$. [2]

3 The chain rule – Part 1

Name: ______________ Class: __________ Date: ____________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
chain rule	链式法则	____________ ____________ ____________

Recall

In the last topic you used the power rule to differentiate powers of x:

- $\dfrac{d}{dx}x^3 = 3x^2$.
- But what about something like $(x^2+1)^3$ –a function **inside** another function?

This sheet gives the rule for that: the chain rule. Each idea has a worked example.

New ideas

■ 1 Composite functions

A **composite function** is one function tucked **inside** another, like $(x^2+1)^3$: the "inside" is x^2+1 and the "outside" is "cube it".

■ 2 The chain rule

To differentiate a composite function, use the **chain rule** 链式法则:

> differentiate the **outside** (keeping the inside as it is), then **multiply** by the derivative of the inside.

Worked example. For $(x^2+1)^3$: the outside gives $3(x^2+1)^2$, and the inside x^2+1 has derivative $2x$, so

$$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2 \cdot 2x.$$

■ 3 Another example

Worked example. For $(3x+1)^4$: outside $\rightarrow 4(3x+1)^3$; inside $3x+1$ has derivative 3; so the answer is $4(3x+1)^3 \cdot 3 = 12(3x+1)^3$.

■ **4 Why "chain"?**

The rule links the rates together in a chain: how fast the outside changes with the inside, times how fast the inside changes with x.

Watch out: don't forget to multiply by the derivative of the **inside**. Writing just $3(x^2+1)^2$ (and forgetting the $\times 2x$) is the most common chain-rule mistake.

Practice

Try these in order. The methods are all above.

2.1 State, in words, what the chain rule tells you to do. [2]

2.2 For $(x^2+1)^3$, state the "inside" function and its derivative. [2]

2.3 Differentiate $(2x+5)^3$. [3]

2.4 Differentiate $(x^2+4)^2$. [3]

2.5 State the most common mistake when using the chain rule. [1]

3 The second derivative and inverse functions – Part 2

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)		
second derivative	二阶导数			

Recall

In Part A you met the chain rule. Now two more ideas:

- You can differentiate a derivative again.
- Some functions are the ”reverse” of others (inverses).

Each idea has a worked example before the Practice.

New ideas

■ 1 The second derivative

If you differentiate a function and then differentiate **again**, you get the **second derivative** 二阶导数, written $f''(x)$.

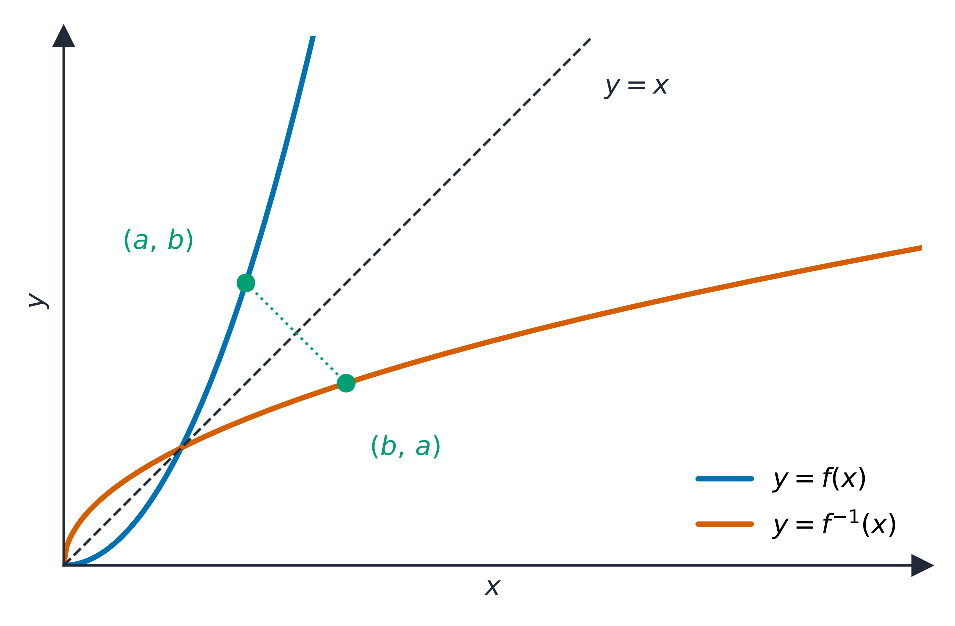
Worked example. For $f(x) = x^3$: the first derivative is $f'(x) = 3x^2$, and differentiating again gives $f''(x) = 6x$.

■ 2 What it means

The first derivative is the **rate of change** (like velocity); the second derivative is the **rate of change of that rate** (like acceleration). It tells you whether the slope is increasing or decreasing.

■ 3 Inverse functions

An **inverse function** reverses a function. Its graph is the reflection of the original in the line $y = x$.



■ 4 Building up

Worked example. For $f(x) = x^2 + 3x$: $f'(x) = 2x + 3$, and $f''(x) = 2$ (a constant, so the slope changes at a steady rate).

Watch out: the second derivative is found by differentiating **twice** –first get $f'(x)$, then differentiate that. It is the derivative of the derivative, not the square of the derivative.

Practice

Try these in order. The methods are all above.

2.1 State what the second derivative is. [1]

2.2 For $f(x) = x^4$, find $f'(x)$ and then $f''(x)$. [3]

2.3 For position over time, state what the first and second derivatives represent. [2]

2.4 State how the graph of an inverse function relates to the original. [2]

2.5 For $f(x) = x^2$, find $f''(x)$. [2]

4 Motion - velocity and acceleration – Part 1

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	**中文**	**Write it 3 times** (English)		
velocity	速度			
acceleration	加速度			

Recall

The derivative is a rate of change –and motion is all about rates:

- Velocity is how fast your position changes.
- Acceleration is how fast your velocity changes.
- Calculus connects all three.

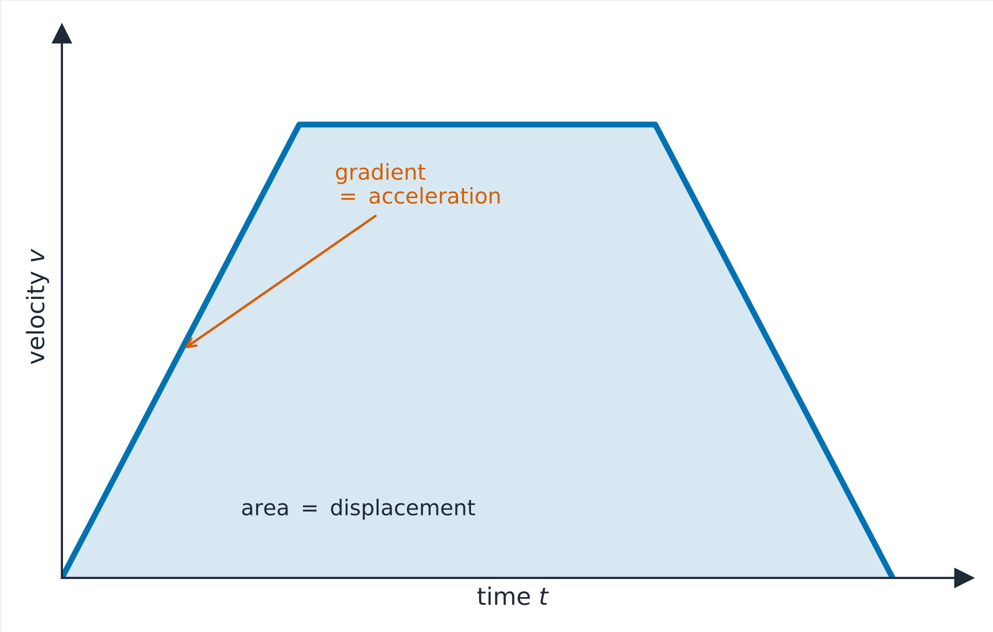
This sheet lines up motion with derivatives. Each idea has a worked example.

New ideas

■ 1 Position, velocity, acceleration

For an object moving along a line:

- **velocity** 速度 is the derivative of position (how fast position changes),
- **acceleration** 加速度 is the derivative of velocity (how fast velocity changes).



■ **2 From position to velocity**

Worked example. If position is $s(t) = t^2$, then velocity is $s'(t) = 2t$. At $t = 3$ the velocity is $2 \times 3 = 6$.

■ **3 From velocity to acceleration**

Worked example. With velocity $v(t) = 2t$, the acceleration is $v'(t) = 2$ –a constant.

■ **4 Direction from the sign**

A **positive** velocity means moving in the positive direction; a **negative** velocity means moving the other way. The object is momentarily at rest when velocity is 0.

Watch out: velocity is the derivative of **position**, and acceleration is the derivative of **velocity** –so acceleration is the derivative of position **twice** (the second derivative).

Practice

Try these in order. The methods are all above.

2.1 State how velocity is related to position, and how acceleration is related to velocity.[2]

2.2 Position is $s(t) = t^2$. Find the velocity $s'(t)$. [2]

2.3 Using that velocity, find the velocity at $t = 5$. [2]

2.4 For velocity $v(t) = 3t$, find the acceleration. [2]

2.5 State what a velocity of 0 tells you about the object's motion. [1]

4 Related rates and linear approximation – Part 2

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
Related rates	相关变化率	____________ ____________ ____________

Recall

In Part A you used derivatives for motion. Two more uses:

- When one thing changes, a connected thing changes too.
- A tangent line can estimate values near a known point.

Each idea has a worked example before the Practice.

New ideas

1 Related rates

Related rates 相关变化率 problems appear when two quantities change together over time. If you know how fast one changes, you can find how fast the other does.

Worked example. A square's side grows at 2 cm/s. Since area $A = s^2$, the area's rate of change is $\frac{dA}{dt} = 2s \cdot \frac{ds}{dt}$. When $s = 5$, $\frac{dA}{dt} = 2(5)(2) = 20$ cm^2/s.

2 Linear approximation

The tangent line at a point stays close to the curve nearby, so it gives a quick **estimate** of the function's value.

3 Using the tangent

Near a known point, $f(x) \approx f(a) + f'(a)(x - a)$ –follow the tangent line a small step.

Worked example. For $f(x) = x^2$ near $x = 3$: $f(3) = 9$ and $f'(3) = 6$, so $f(3.1) \approx 9 + 6(0.1) = 9.6$ (the true value is 9.61).

■ **4 Why it works**

Because a smooth curve looks almost straight when you zoom in, the tangent is an excellent approximation close to the point.

Watch out: a tangent-line estimate is only good **close** to the point of contact. Far from it, the curve bends away from the tangent and the approximation gets worse.

Practice

Try these in order. The methods are all above.

2.1 State what a related-rates problem involves. [2]

2.2 A square's side grows at 3 cm/s. Using $\frac{dA}{dt} = 2s\frac{ds}{dt}$, find the rate the area grows when $s = 4$. [3]

2.3 State what a linear approximation uses to estimate a function's value. [1]

2.4 For $f(x) = x^2$ near $x = 2$ ($f(2) = 4$, $f'(2) = 4$), estimate $f(2.1)$. [3]

2.5 State when a tangent-line approximation stops being accurate. [1]

5 Increasing, decreasing, and turning points – Part 1

Name: ______________ Class: __________ Date: ____________

Recall

The derivative gives the slope of a curve –and the slope tells you a lot:

- Where a graph goes uphill, the slope is positive.
- At the very top of a hill, the graph is momentarily flat.
- Businesses want to find the ”best” (maximum or minimum) point.

This sheet uses the derivative to find high and low points. Each idea has a worked example.

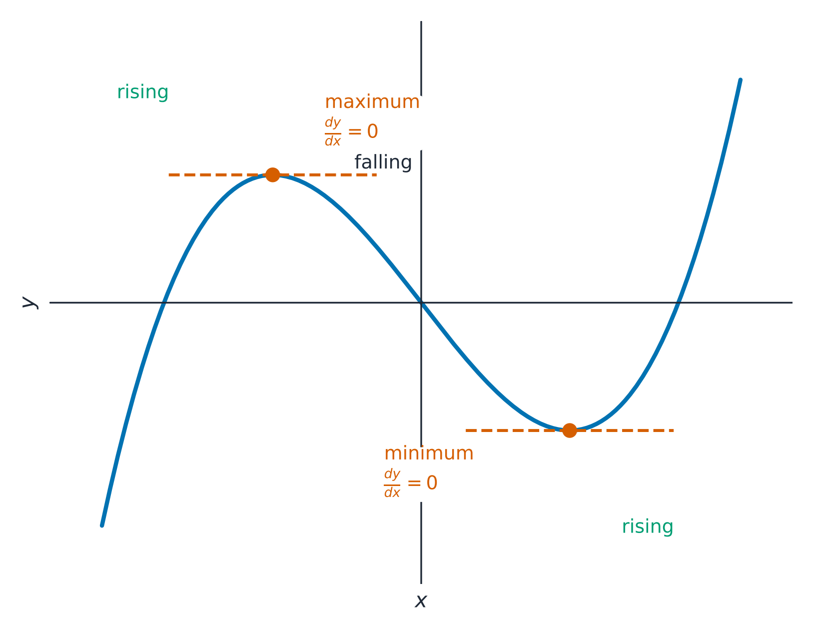
New ideas

1 Increasing and decreasing

- Where the derivative $f'(x) > 0$, the function is **increasing** (going uphill).
- Where $f'(x) < 0$, it is **decreasing** (going downhill).

2 Stationary points

Where $f'(x) = 0$, the curve is momentarily **flat** –a **stationary point** (a peak or a valley).



■ 3 Finding a maximum or minimum

To find high and low points: differentiate, set $f'(x) = 0$, and solve for x.

Worked example. For $f(x) = x^2 - 4x$: $f'(x) = 2x - 4$. Setting $2x - 4 = 0$ gives $x = 2$ –the minimum point of this parabola.

■ 4 Peak or valley?

Check the slope just before and after: if it changes from positive to negative, it is a **maximum**; from negative to positive, a **minimum**.

Watch out: a stationary point ($f' = 0$) can be a maximum **or** a minimum (or neither). Finding where $f'(x) = 0$ locates the point; you still have to check which kind it is.

Practice

Try these in order. The methods are all above.

2.1 State what the sign of $f'(x)$ tells you about the graph. [2]

2.2 State what is true about the derivative at a stationary point. [1]

2.3 For $f(x) = x^2 - 6x$, find $f'(x)$ and the x-value where it is zero. [3]

2.4 For $f(x) = x^2 + 2x$, find the x-value of the stationary point. [2]

2.5 State how you decide whether a stationary point is a maximum or a minimum. [2]

5 Concavity and the shape of a curve – Part 2

Name: ______________ Class: __________ Date: ____________

Recall

In Part A you found peaks and valleys with the first derivative. Now the **second** derivative describes the curve's bend:

- A curve can bend upward (like a smile) or downward (like a frown).
- The second derivative tells you which.

Each idea has a worked example before the Practice.

New ideas

1 Concavity

The way a curve bends is its **concavity**:

- $f''(x) > 0$: **concave up** (a smile / valley shape),
- $f''(x) < 0$: **concave down** (a frown / hill shape).

2 Points of inflection

Where the curve **changes** its bend (from concave up to concave down or the reverse) is a **point of inflection** –there $f''(x) = 0$.

3 The second-derivative test

At a stationary point ($f' = 0$): if $f'' > 0$ it is a **minimum** (valley), and if $f'' < 0$ it is a **maximum** (hill).

Worked example. For $f(x) = x^2 - 4x$: $f'(x) = 2x - 4 = 0$ at $x = 2$, and $f''(x) = 2 > 0$, so $x = 2$ is a **minimum**.

4 Putting it together

The first derivative finds where the peaks and valleys are; the second derivative tells you which is which and how the curve bends.

Watch out: concave up is a ”smile” (holds water) and concave down is a ”frown” (spills water). A concave-up curve can still be going downhill –concavity is about the **bend**, not the slope.

Practice

Try these in order. The methods are all above.

2.1 State what $f''(x) > 0$ and $f''(x) < 0$ tell you about a curve's shape. [2]

2.2 State what a point of inflection is. [2]

2.3 For $f(x) = x^2 + 6x$: find $f'(x)$, the stationary point, and use $f''(x)$ to say if it is a max or min. [3]

2.4 State the second-derivative test for a minimum. [1]

2.5 Explain why ”concave up” does not always mean the curve is going uphill. [2]

6 The area under a curve – Part 1

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	中文	Write it 3 times (English)
Riemann sum	黎曼和	

Recall

You can find the area of simple shapes:

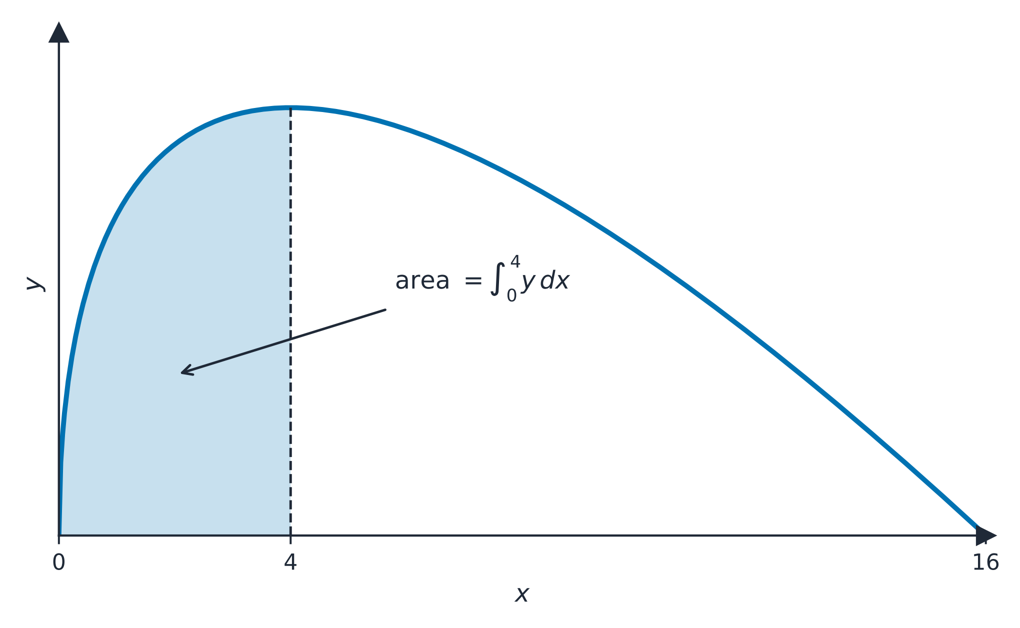
- The area of a rectangle is width × height.
- But how do you find the area under a **curve**, which has no straight edges?

This sheet introduces the idea behind **integration**. Each idea has a worked example.

New ideas

■ 1 Area under a curve

The **area under a curve** (between the curve and the x-axis) is a useful quantity – for example, the area under a velocity–time graph is the distance travelled.



■ 2 Approximating with rectangles

You can estimate that area by covering it with thin **rectangles** and adding their areas –a **Riemann sum** 黎曼和.

■ 3 More rectangles, better estimate

The thinner the rectangles (the more you use), the closer the total gets to the true area.

Worked example. To estimate the area under $y = x$ from 0 to 4 with rectangles: the exact area is a triangle, $\frac{1}{2} \times 4 \times 4 = 8$, and more rectangles get closer and closer to this.

■ 4 The exact area

Making the rectangles infinitely thin gives the **exact** area –this exact value is the **integral**.

Watch out: a Riemann sum with a few rectangles is only an **estimate**. The exact area (the integral) is the limit as the rectangles become infinitely thin and infinitely many.

Practice

Try these in order. The methods are all above.

2.1 State what ”the area under a curve” means. [2]

2.2 State how a Riemann sum estimates that area. [2]

2.3 State what happens to the estimate as you use more, thinner rectangles. [1]

2.4 Find the exact area under $y = x$ from 0 to 6 (it is a triangle). [3]

2.5 State what the area under a velocity–time graph represents. [1]

6 Integration and the power rule – Part 2

Name: ______________ Class: __________ Date: ____________

Recall

In Part A you saw that the integral is the exact area under a curve. Now compute it:

- Integration is the **reverse** of differentiation.
- There is a simple rule for powers of x, just like the power rule for derivatives.

Each idea has a worked example before the Practice.

New ideas

1 Integration undoes differentiation

Integration finds a function whose derivative is the one you started with –it is the reverse of differentiating. This reverse is written with the sign $\int$.

2 The power rule for integration

To integrate a power of x, **raise the power by one and divide by the new power**:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Worked example. $\int x^2 \, dx = \dfrac{x^3}{3} + C$ (check: differentiating $\frac{x^3}{3}$ gives x^2).

3 The "+ C"

Because many functions share the same derivative (they differ by a constant), we add a constant C when integrating.

4 Definite integral gives area

A **definite integral** (with limits) gives the exact area under the curve between two x-values.

Worked example. The area under $y = x$ from 0 to 2 is $\left[\frac{x^2}{2}\right]_0^2 = \frac{4}{2} - 0 = 2$.

Watch out: integration **raises** the power and **divides** –the exact opposite of differentiation, which lowers the power and multiplies. Mixing them up is a common slip.

Practice

Try these in order. The methods are all above.

2.1 State how integration relates to differentiation. [1]

2.2 Find $\int x^3 \, dx$. [2]

2.3 Find $\int x^4 \, dx$. [2]

2.4 State why we add "$+C$" when integrating. [2]

2.5 Find the area under $y = x$ from 0 to 4 using $\left[\frac{x^2}{2}\right]_0^4$. [3]

7 Differential equations and slope fields – Part 1

Name: ______________ Class: __________ Date: ____________

Recall

You have solved ordinary equations to find a number:

- An equation like $2x + 1 = 7$ has the solution $x = 3$.
- But some equations describe how a quantity **changes** –they involve a derivative.

This sheet introduces differential equations. Each idea has a worked example.

New ideas

■ 1 What a differential equation is

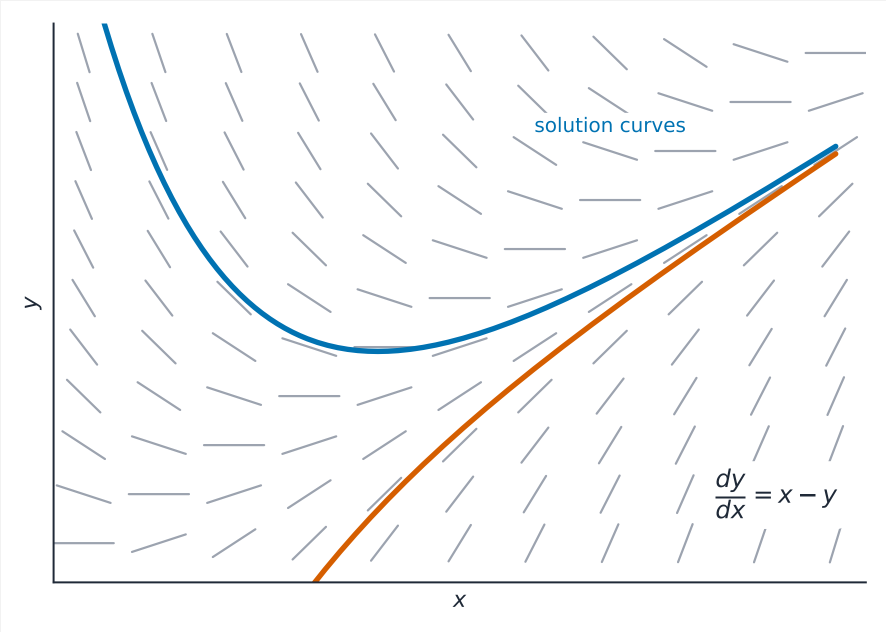
A **differential equation** involves a **derivative** –it tells you the **rate of change**, not the value directly. For example $\frac{dy}{dx} = 2x$ says "the slope at each point is $2x$".

■ 2 Its solution is a function

Solving it means finding the **function** y whose derivative matches. For $\frac{dy}{dx} = 2x$, one solution is $y = x^2$ (its derivative is $2x$).

■ 3 Slope fields

A **slope field** draws a tiny line at many points showing the slope there. It lets you see the shape of the solutions without solving.



■ 4 A family of solutions

There are usually **many** solutions differing by a constant (like $y = x^2$, $y = x^2 + 1$, $\cdots$) –a whole family that follows the slope field.

Watch out: a differential equation gives the **slope** (dy/dx), not the function itself. You must integrate (or otherwise solve) to recover the actual function y.

Practice

Try these in order. The methods are all above.

2.1 State what makes an equation a ”differential equation”. [2]

2.2 State what it means to solve a differential equation. [2]

2.3 For $\dfrac{dy}{dx} = 2x$, verify that $y = x^2$ is a solution. [2]

2.4 State what a slope field shows. [2]

2.5 State why a differential equation usually has many solutions. [1]

7 Solving simple differential equations – Part 2

Name: ____________ Class: ________ Date: __________

Recall

In Part A you met differential equations. Now solve simple ones:

- Solving means finding the function from its rate of change.
- One special kind describes things that grow by a fixed percentage –like money or populations.

Each idea has a worked example before the Practice.

New ideas

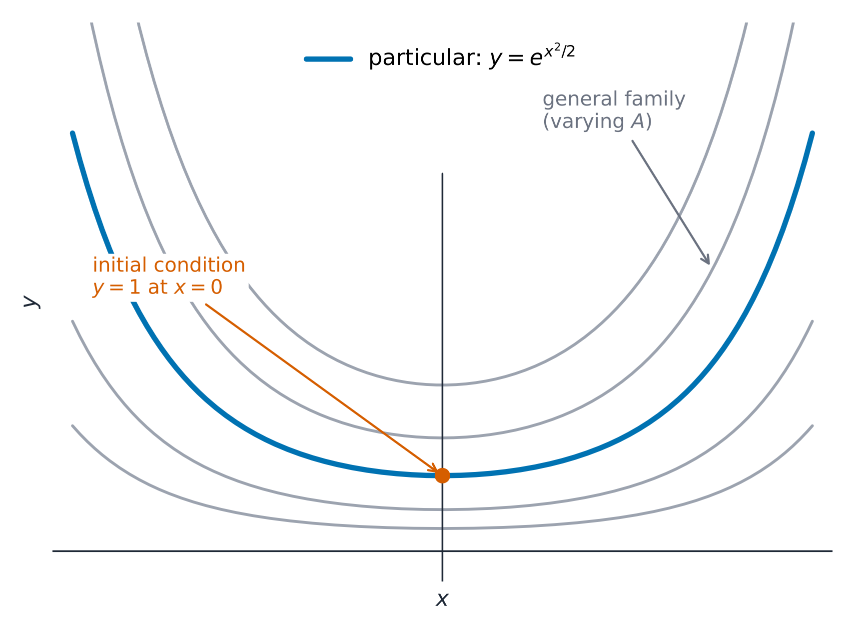
1 Solving by integrating

If $\dfrac{dy}{dx}$ is given, integrate to find y.

Worked example. $\dfrac{dy}{dx} = 3x^2$ integrates to $y = x^3 + C$.

2 A family of curves

The constant C gives a **family** of solution curves; one extra fact (a known point) picks out the right one.



■ 3 Exponential growth

A very common equation is $\frac{dy}{dt} = ky$ –"the rate of growth is proportional to how much there is". Its solution is exponential growth, $y = Ce^{kt}$.

■ 4 Where it appears

This models populations, money earning interest, and radioactive decay (with a negative k) –anything that grows or shrinks by a fixed proportion.

Watch out: exponential growth ($\frac{dy}{dt} = ky$) means the quantity grows **faster and faster** as it gets bigger, because the rate depends on the current amount. It is not steady, straight-line growth.

Practice

Try these in order. The methods are all above.

2.1 State how you solve $\frac{dy}{dx} = f(x)$ for y. [1]

2.2 Solve $\dfrac{dy}{dx} = 2x$. [2]

2.3 Solve $\dfrac{dy}{dx} = 4x^3$. [2]

2.4 State the type of growth described by $\dfrac{dy}{dt} = ky$. [1]

2.5 Give two real situations modelled by exponential growth or decay. [2]

8 Area between two curves – Part 1

Name: ______________ Class: _________ Date: ____________

Recall

You have found the area under a single curve. Now the area **between** two curves:

- Two curves can enclose a region between them.
- Finding that area is a common use of integration.

Each idea has a worked example before the Practice.

New ideas

1 Area between two curves

The area between an upper curve and a lower curve is the integral of the **difference**:

$$\text{area} = \int (\text{top} - \text{bottom})\, dx.$$

2 Why subtract

The area under the top curve minus the area under the bottom curve leaves exactly the region between them.

3 A worked example

Worked example. Between $y = 4$ (top) and $y = x^2$ (bottom), where they meet at $x = -2$ and $x = 2$, the area is $\int_{-2}^{2}(4 - x^2)\, dx$ –the constant 4 line minus the parabola.

4 Getting the limits

The two x-values where the curves **cross** are the limits of the integral –set the two functions equal to find them.

Watch out: always subtract **top minus bottom** (the higher curve minus the lower). Subtracting the wrong way round gives a **negative** area, which is a sign you have them reversed.

Practice

Try these in order. The methods are all above.

2.1 Write the formula for the area between an upper and a lower curve. [2]

2.2 Explain why you subtract the two curves. [2]

2.3 For the region between $y = 6$ and $y = x^2$, state what you would integrate. [2]

2.4 State how you find the limits of the integral. [2]

2.5 State what a negative area tells you about your set-up. [1]

8 Volumes and average value – Part 2

Name: ______________ Class: _________ Date: ____________

Recall

In Part A you found areas with integration. Integration does even more:

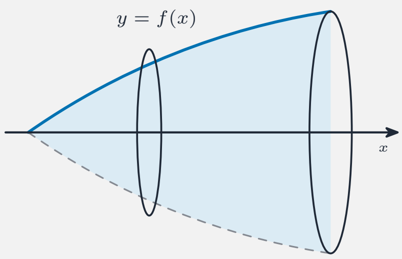
- Spin a flat region around a line and it sweeps out a 3-D solid.
- Integration can find that solid's volume, and the average value of a function.

Each idea has a worked example before the Practice.

New ideas

1 Volumes of revolution

If you spin a region around an axis, it sweeps out a **solid of revolution** –like spinning a rectangle to make a cylinder, or a curve to make a vase shape.



rotate the region around the x-axis to sweep a solid

2 Slicing into discs

Integration finds the volume by adding up many thin **discs**, each a circle of area πr^2 times a tiny thickness.

3 Average value

The **average value** of a function over an interval is its integral divided by the width of the interval –the "typical" height of the curve.

Worked example. The average value of $y = x$ from 0 to 4 is $\dfrac{1}{4}\int_0^4 x\,dx = \dfrac{1}{4} \times 8 = 2.$

4 Integration as "adding up"

The common thread: integration **adds up infinitely many tiny pieces** –tiny areas, tiny discs, or tiny contributions to an average.

Watch out: integration is best understood as ”adding up many tiny pieces” –tiny rectangles for area, tiny discs for volume. Whenever a quantity is a sum of infinitely many small parts, an integral finds it.

Practice

Try these in order. The methods are all above.

2.1 State what a solid of revolution is. [2]

2.2 State the shape you get by spinning a rectangle around one of its sides. [1]

2.3 State how integration finds the volume of a solid of revolution. [2]

2.4 Find the average value of $y = x$ from 0 to 6 (using $\frac{1}{6}\int_0^6 x\,dx$ with $\int_0^6 x\,dx = 18$). [3]

2.5 In one sentence, state what integration does in all these cases. [1]

9 Parametric and vector functions – Part 1

Name: ______________ Class: __________ Date: ____________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	**中文**	**Write it 3 times** (English)		
parametric function	参数方程			
vector	向量			

Recall

So far a function gave one output for each input, like $y = f(x)$:

- But a moving object needs both its across (x) and up (y) positions.
- Both can be given in terms of time.

This sheet introduces parametric and vector functions. Each idea has a worked example.

New ideas

1 Parametric functions

A **parametric function** 参数方程 gives both coordinates in terms of a third variable (a **parameter**), usually time t: $\big(x(t), y(t)\big)$. As t runs, the point traces a path.

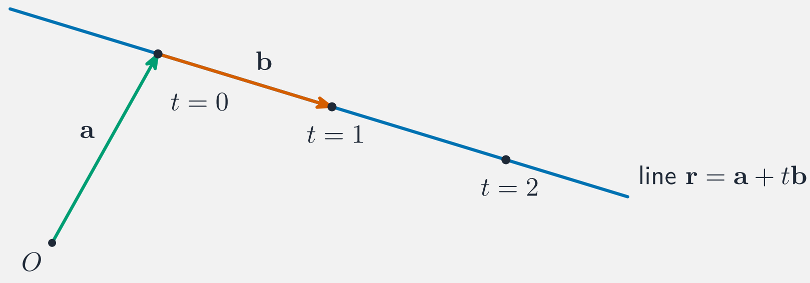
Worked example. $x(t) = t$, $y(t) = t^2$: at $t = 0$ the point is $(0, 0)$; at $t = 1$, $(1, 1)$; at $t = 2$, $(2, 4)$ –tracing a parabola.

2 Direction of travel

Because t is like time, the path has a **direction** –the order in which it is drawn as t increases.

3 Vectors

A **vector** 向量 has magnitude and direction, written by components $\langle a, b\rangle$. A **vector-valued function** packages a moving point as $\vec{r}(t) = \langle x(t), y(t)\rangle$.



4 Same idea, two forms

A parametric function and a vector-valued function carry the **same** information –the position of a moving point at each time t.

Watch out: in a parametric curve, y is **not** given directly in terms of x –both are given in terms of t. To find the ordinary y-vs-x shape you eliminate t.

Practice

Try these in order. The methods are all above.

2.1 State what a parametric function gives you. [2]

2.2 For $x(t) = t$, $y(t) = t^2$, find the point when $t = 3$. [2]

2.3 For $x(t) = 2t$, $y(t) = t + 1$, find the point when $t = 0$ and when $t = 2$. [2]

2.4 State the two things a vector has. [2]

2.5 State what a parametric and a vector-valued function have in common. [1]

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9 Polar coordinates – Part 2

Name: ____________ Class: ________ Date: __________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	**中文**	**Write it 3 times** (English)		
Polar coordinates	极坐标	____________	____________	____________

Recall

You locate points on a grid with (x, y) coordinates:

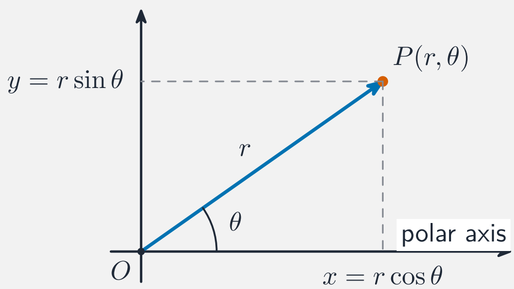
- x is how far across, y is how far up.
- But you could also describe a point by its **distance** and **direction** from the centre.

Each idea has a worked example before the Practice.

New ideas

■ 1 Polar coordinates

Polar coordinates 极坐标 locate a point by its distance r from the origin and its angle θ from the positive x-axis –written (r, θ).



■ 2 Converting to x and y

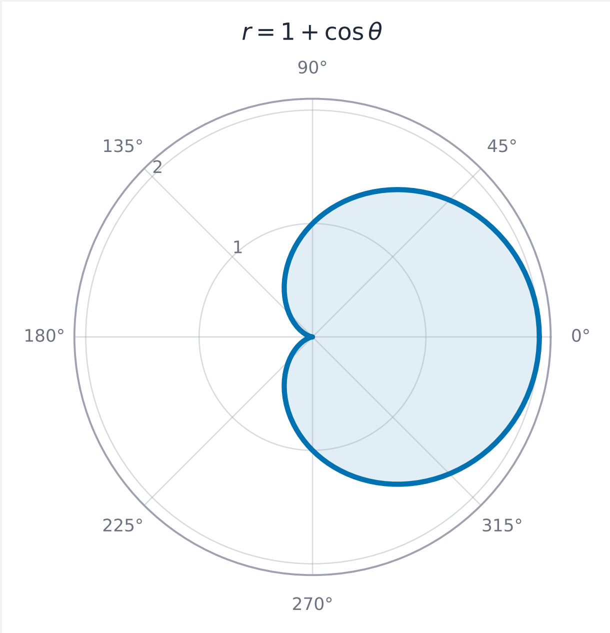
To convert polar to the usual coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Worked example. The polar point $(2, 0°)$ is $x = 2\cos 0° = 2$, $y = 2\sin 0° = 0$ –the point $(2, 0)$.

■ 3 Polar curves

Letting r depend on θ traces a **polar curve** –often beautiful shapes like circles, spirals, and flower petals.



■ 4 Why use polar

Polar coordinates make round and rotating shapes much simpler to describe than x–y coordinates do.

Watch out: in polar coordinates the first number is the **distance** r and the second is the **angle** θ –not an x and a y. Reading (2, 90°) as "across 2, up 90" is wrong; it means "distance 2, angle 90°".

Practice

Try these in order. The methods are all above.

2.1 State how polar coordinates locate a point. [2]

2.2 Write the formulas that convert polar (r, θ) to x and y. [2]

2.3 Convert the polar point $(3, 0°)$ to x–y coordinates. [2]

2.4 Convert the polar point $(4, 90°)$ to x–y coordinates ($\cos 90° = 0$, $\sin 90° = 1$). [3]

2.5 State one advantage of polar coordinates. [1]

10 Infinite series – Part 1

Name: ______________ Class: __________ Date: ____________

Vocabulary 词汇

Write each word three times. 把每个单词抄写三遍。

English	**中文**	**Write it 3 times** (English)		
sequence	数列			
series	级数			
geometric series	等比级数			

Recall

You met geometric sequences earlier –each term multiplies by a fixed ratio:

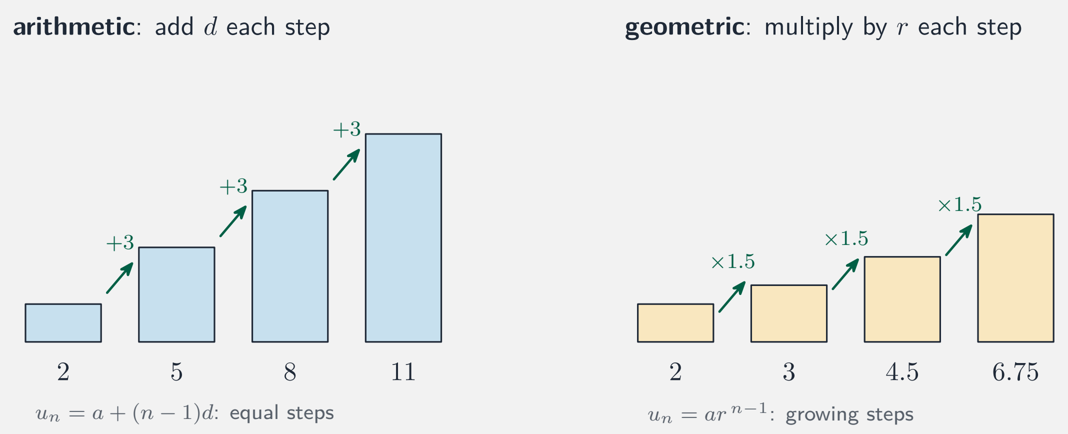
- $2, 4, 8, 16, \ldots$ doubles each time.
- What if you add up **all** the terms? Can an infinite sum have a finite answer?

This sheet introduces infinite series. Each idea has a worked example.

New ideas

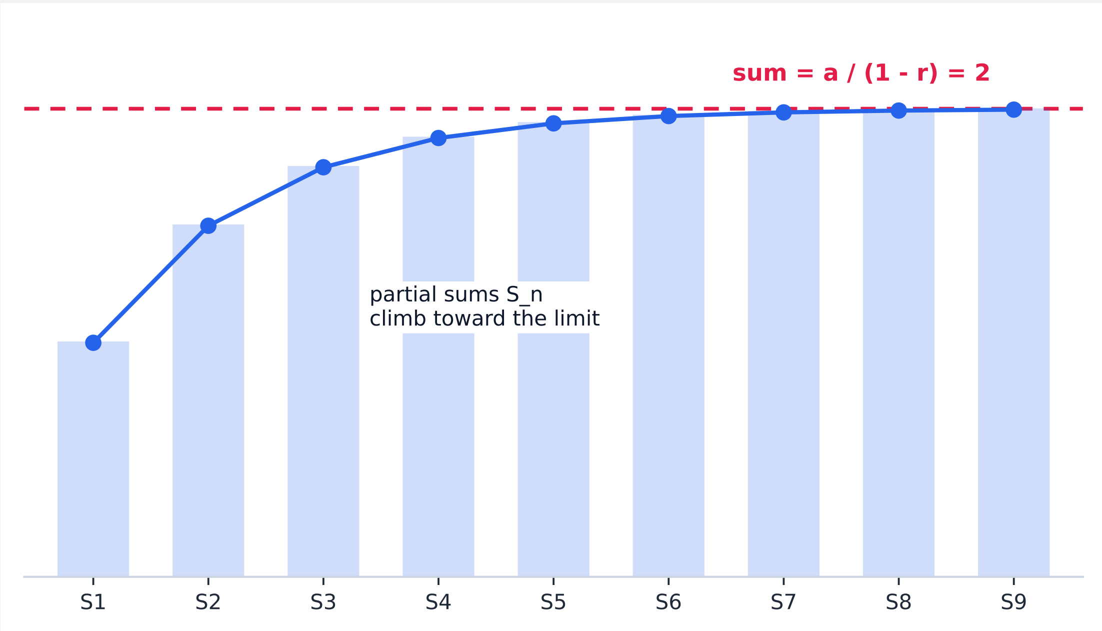
■ 1 Sequence versus series

A **sequence** 数列 is a **list** of terms; a **series** 级数 is the **sum** of those terms.



2 Geometric series

A **geometric series** 等比级数 adds terms that each multiply by a ratio r: $a + ar + ar^2 + \cdots$.



3 Infinite sum

If $|r| < 1$, the terms shrink toward zero and the infinite sum **converges** to a finite value:

$$S = \frac{a}{1-r}.$$

Worked example. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ has $a = 1$, $r = \frac{1}{2}$, so $S = \dfrac{1}{1-\frac{1}{2}} = 2$.

4 Converge or diverge

If $|r| \geq 1$ the terms do not shrink, so the sum grows without limit –it **diverges** (no finite answer).

Watch out: an infinite sum only has a finite value when the terms **shrink fast enough** ($|r| < 1$ for a geometric series). If the terms stay the same size or grow, the sum diverges.

Practice

Try these in order. The methods are all above.

2.1 State the difference between a sequence and a series. [2]

2.2 Write the formula for the sum of an infinite geometric series with $|r| < 1$. [1]

2.3 Find the sum of $1 + \frac{1}{3} + \frac{1}{9} + \cdots$ (with $a = 1$, $r = \frac{1}{3}$). [3]

2.4 Find the sum of $4 + 2 + 1 + \frac{1}{2} + \cdots$ (with $a = 4$, $r = \frac{1}{2}$). [2]

2.5 State whether $1 + 2 + 4 + 8 + \cdots$ converges, and why. [2]

10 Power and Taylor series – Part 2

Name: ______________ Class: ________ Date: ____________

Recall

In Part A you added up infinitely many terms. Now use that to **approximate functions**:

- A complicated function can be replaced by a simple polynomial.
- Adding more terms makes the approximation better.

Each idea has a worked example before the Practice.

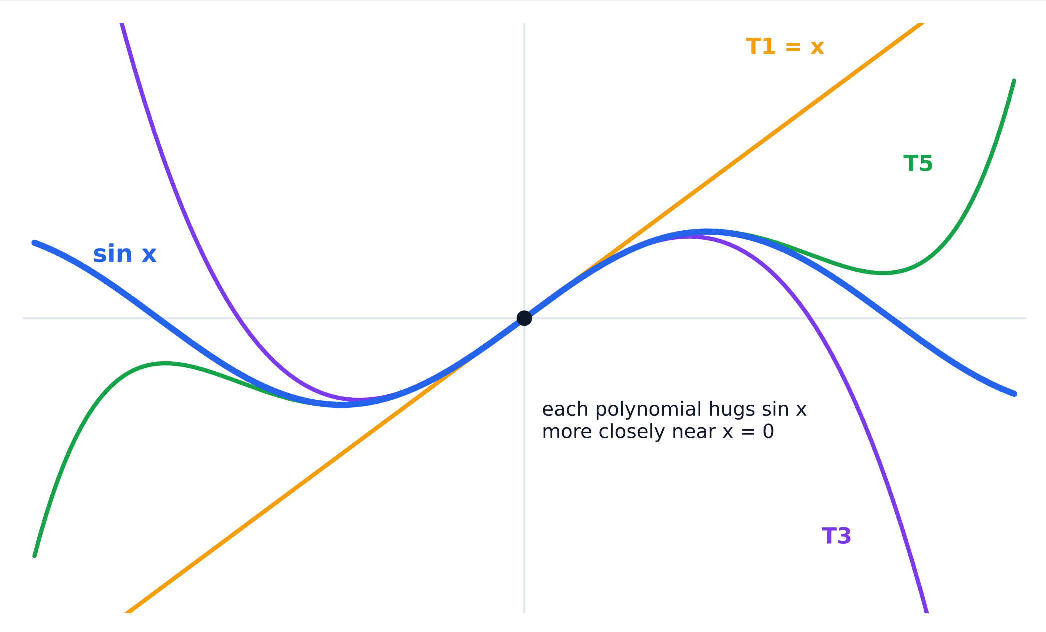
New ideas

1 Power series

A **power series** adds up powers of x: $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$. It behaves like an "infinitely long polynomial".

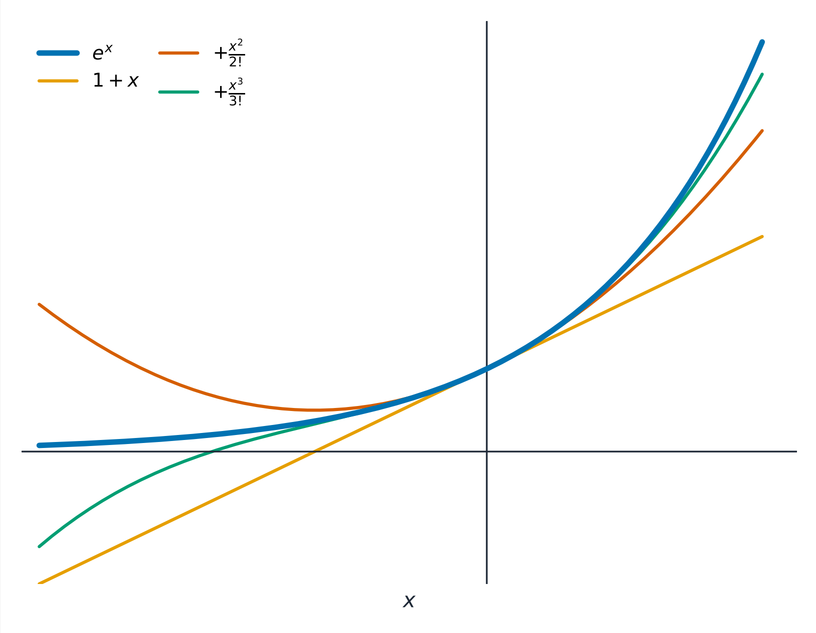
2 Taylor and Maclaurin series

A **Taylor series** (a **Maclaurin series** when centred at 0) builds a power series that matches a function near a point –approximating it by a polynomial.



3 More terms, better fit

The more terms you keep, the more closely the polynomial hugs the real function.



■ 4 Why it matters

Polynomials are easy to add, differentiate, and integrate, so replacing a hard function with its series makes calculations far simpler –calculators use this idea.

Watch out: a Taylor approximation is most accurate **near the point** it is built around. Far from that point, you need more terms to keep it accurate.

Practice

Try these in order. The methods are all above.

2.1 State what a power series is. [2]

2.2 State what a Taylor (or Maclaurin) series does. [2]

2.3 State what happens to the approximation as you keep more terms. [1]

2.4 Explain why replacing a function with its power series is useful. [2]

2.5 State where a Taylor approximation is most accurate. [1]