Part B (AB): Graphing calculator not allowed

Question 5 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Two particles, H and J, are moving along the x-axis. For $0 \le t \le 5$, the position of particle H at time t is given by $x_H(t) = e^{t^2-4t}$ and the velocity of particle J at time t is given by $v_J(t) = 2t\left(t^2-1\right)^3$.

	Model Solution	Scoring	
A	Find the velocity of particle H at time $t = 1$. Show the work that leads to your answer.		
	$x_H{}'(t) = v_H(t) = (2t-4)e^{t^2-4t}$	Considers $x_H{}'$	**Point 1 (P1)**
	$x_H{}'(1) = v_H(1) = -2e^{-3}$	Answer	**Point 2 (P2)**

Scoring Notes for Part A

- **P1** can be earned by presenting $x_H{}'(t)$, $x_H{}'(1)$, $x'(t)$, $x'(1)$, $(2t-4)e^{t^2-4t}$, or $(2\cdot 1-4)e^{1^2-4\cdot 1}$.
- An unsupported answer of $-2e^{-3}$ earns **P2** but not **P1**.

B	During what open intervals of time t, for $0 < t < 5$, are particles H and J moving in opposite directions? Give a reason for your answer.		
	From part A, $x_H'(t) = v_H(t) = (2t-4)e^{t^2-4t}$.	Considers sign of $x_H'(t)$ or $v_J(t)$	**Point 3 (P3)**
	$x_H'(t) = (2t-4)e^{t^2-4t} = 0 \Rightarrow t = 2$	Analysis for one particle	**Point 4 (P4)**
	$x_H'(t) < 0$ for $0 < t < 2$, and $x_H'(t) > 0$ for $2 < t < 5$.	Answer with reason	**Point 5 (P5)**
	Thus, particle H is moving to the left for $0 < t < 2$ and moving to the right for $2 < t < 5$.		
	$v_J(t) = 2t\left(t^2-1\right)^3 = 0$ for $0 < t < 5 \Rightarrow t = 1$		
	$v_J(t) < 0$ for $0 < t < 1$, and $v_J(t) > 0$ for $1 < t < 5$.		
	Thus, particle J is moving to the left for $0 < t < 1$ and moving to the right for $1 < t < 5$.		
	Therefore, particles H and J are moving in opposite directions for $1 < t < 2$.		

Scoring Notes for Part B

- To earn **P3**, a response can do one of the following:
  - Set $x_H'(t) = 0$, $v_H(t) = 0$, or $(2t-4)e^{t^2-4t} = 0$
  - Set $v_J(t) = 0$ or $2t\left(t^2-1\right)^3 = 0$
  - Identify $t = 2$ for particle H and no other values in the interval $0 < t < 5$
  - Identify $t = 1$ for particle J and no other values in the interval $0 < t < 5$
  - Identify the interval $1 < t < 2$
- To earn **P4**, a response can provide an analysis of signs of velocity or direction of motion on the interval $0 < t < 5$ for either particle H or particle J.
- To be eligible for **P5**, a response must provide correct analyses of signs of velocity or direction of motion on the interval $0 < t < 5$ for both particles.
- Only analysis within the interval $0 < t < 5$ will be considered in scoring.

C It can be shown that $v_J{}'(2) > 0$. Is the speed of particle J increasing, decreasing, or neither at time $t = 2$? Give a reason for your answer.

$v_J(2) > 0$ and $v_J{}'(2) > 0$. Because $v_J(2)$ and $v_J{}'(2)$ have the same sign, the speed of particle J is increasing at $t = 2$.	Answer with reason	**Point 6 (P6)**

Scoring Notes for Part C

- An evaluation of $v_J(2)$ is not necessary, but if a value is presented, it must be correct. The correct value is $v_J(2) = 108$.
- An evaluation of $v_J{}'(2)$ is not necessary, but if a value is presented, it must be correct. The correct value is $v_J{}'(2) = 486$.
- A response can either import the analysis for the sign of $v_J(2)$ from part B or restart.
- A response that stated "$v_J(t) > 0$ for $1 < t < 5$" in part B does not need to restate $v_J(2) > 0$ and earns **P6** for "$v_J(2)$ and $v_J{}'(2)$ have the same sign, so the speed is increasing."

D Particle J is at position $x = 7$ at time $t = 0$. Find the position of particle J at time $t = 2$. Show the work that leads to your answer.

$$x_J(2) = x_J(0) + \int_0^2 v_J(t)\,dt = 7 + \int_0^2 2t\left(t^2 - 1\right)^3 dt$$

$$= 7 + \left[\frac{1}{4}\left(t^2 - 1\right)^4\right]_0^2$$

$$= 7 + \frac{1}{4}\left((3)^4 - (-1)^4\right) = 7 + \frac{1}{4}(80) = 27$$

Integrand	**Point 7 (P7)**
Antiderivative	**Point 8 (P8)**
Answer	**Point 9 (P9)**

Scoring Notes for Part D

- To earn **P7**, a response must present an indefinite or definite integral with an integrand of $v_J(t)$ or $2t\left(t^2 - 1\right)^3$. (See below for notes on how to handle a missing differential dt.)
- **P8** is earned for an antiderivative of the form $k\left(t^2 - 1\right)^4$ or equivalent, for $k > 0$. If $k \neq \frac{1}{4}$, then the response is not eligible to earn **P9**.
- A response of $7 + \frac{1}{4}\left((3)^4 - (-1)^4\right)$ or equivalent banks **P9** (i.e., subsequent errors in simplification will not be considered in scoring for **P9**).

  Note: An ambiguous response, such as $7 + \frac{1}{4}\left((3)^4 - (-1)^4\right.$, does not bank **P9** and therefore must go on to resolve the ambiguity with a correct final answer (e.g., $7 + \frac{1}{4}(80)$ or 27) to earn **P9**.

- If the differential dt is missing:
  - Writing $\int_0^2 v_J(t)$ earns **P7** and is eligible to earn **P8** and **P9**.
  - Writing $7 + \int_0^2 v_J(t)$ earns **P7** and is eligible to earn **P8** and **P9**.
  - Writing $\int_0^2 v_J(t) + 7$ introduces an ambiguity for the intended integrand.
    - $\int_0^2 v_J(t) + 7 = \left[\frac{1}{4}\left(t^2 - 1\right)^4\right]_0^2 + 7$ resolves the ambiguity.

      Therefore, this earns **P7** and **P8** and is eligible for **P9**.
    - $\int_0^2 v_J(t) + 7 = \left[\frac{1}{4}\left(t^2 - 1\right)^4 + 7t\right]_0^2$ confirms that an incorrect integrand was used.

      Therefore, this does not earn **P7**, earns **P8**, and is not eligible for **P9**.
    - If the ambiguity is not resolved, this does not earn **P7**, **P8**, or **P9**.
- Alternate solution using u-substitution:

  Let $u = t^2 - 1$, then $du = 2t\ dt$.

  $t = 0 \Rightarrow u = -1$

  $t = 2 \Rightarrow u = 3$

  $$x_J(2) = x_J(0) + \int_0^2 v_J(t)\ dt = 7 + \int_0^2 2t\left(t^2 - 1\right)^3 dt$$

  $$= 7 + \int_{-1}^3 u^3\ du = 7 + \left[\frac{1}{4}u^4\right]_{-1}^3$$

  $$= 7 + \frac{1}{4}\left((3)^4 - (-1)^4\right) = 7 + \frac{1}{4}(80) = 27$$
- Alternate solution using indefinite integral:

  $$\int 2t\left(t^2 - 1\right)^3 dt = \frac{1}{4}\left(t^2 - 1\right)^4 + C$$

  $$x_J(0) = 7 = \frac{1}{4}\left(0^2 - 1\right)^4 + C \Rightarrow C = \frac{27}{4}$$

  $$x_J(t) = \frac{1}{4}\left(t^2 - 1\right)^4 + \frac{27}{4}$$

  $$x_J(2) = \frac{1}{4}\left(2^2 - 1\right)^4 + \frac{27}{4} = \frac{108}{4} = 27$$

Part B (AB): Graphing calculator not allowed

Question 5 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\dfrac{dy}{dx} = \dfrac{-2x}{3 + 4y}$.

Model Solution	Scoring	
(a) There is a point on the curve near $(2, 4)$ with x-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y-coordinate of this point.		
$\left.\dfrac{dy}{dx}\right\|_{(x,\, y)=(2,\, 4)} = \dfrac{-2(2)}{3 + 4(4)} = -\dfrac{4}{19}$	Slope of tangent line	**1 point**
$y \approx 4 - \dfrac{4}{19}(3 - 2) = \dfrac{72}{19}$	Approximation	**1 point**

Scoring notes:

- A response earns the first point for finding $\left.\dfrac{dy}{dx}\right|_{(x,\, y)=(2,\, 4)} = -\dfrac{4}{19}$, even if this is not labeled or used as the slope of a tangent line.
- A response that does not explicitly find the value of $\left.\dfrac{dy}{dx}\right|_{(x,\, y)=(2,\, 4)}$ but uses a slope of $-\dfrac{4}{19}$ in a linear approximation also earns the first point.
- A response that declares $\left.\dfrac{dy}{dx}\right|_{(x,\, y)=(2,\, 4)}$ equal to any nonzero value other than $-\dfrac{4}{19}$ does not earn the first point but is eligible for the second point for a linear approximation at $x = 3$ through the point $(2, 4)$ with a slope of the declared value.
  - The second point cannot be earned with a linear approximation using a slope other than $-\dfrac{4}{19}$ if that slope has not been declared to be the value of $\left.\dfrac{dy}{dx}\right|_{(x,\, y)=(2,\, 4)}$.
- The second point cannot be earned for an approximation at any value of x other than 3.
- A response does not have to write the tangent line equation but must clearly demonstrate its use at $x = 3$ in finding the requested approximation in order to earn both points.
- The minimal work required to earn both points is $4 - \dfrac{4}{19}(3 - 2)$.

Total for part (a) — 2 points

(b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

$\frac{dy}{dx} = \frac{-2x}{3 + 4y} = 0 \Rightarrow x = 0$	Considers $\frac{dy}{dx} = 0$	**1 point**
And so, if the horizontal line $y = 1$ is tangent to the curve, the point of tangency must be $(0, 1)$. However, the point $(0, 1)$ is not on the curve, because $0^2 + 3 \cdot 1 + 2 \cdot 1^2 = 5 \neq 48$. Therefore, the horizontal line $y = 1$ is not tangent to the curve.	Answer with reason	**1 point**

Scoring notes:

- The first point can be earned with a response of $\frac{dy}{dx} = 0$, $-2x = 0$, or $x = 0$, OR by identifying the point $(0, 1)$.
- To earn the second point a response must provide a reason that the line $y = 1$ is not tangent to the curve. Merely stating "$(0, 1)$ does not lie on the curve" is not sufficient to earn the second point.
- Alternate solution:

  If $y = 1$, then $x^2 + 3 \cdot 1 + 2 \cdot 1^2 = 48 \Rightarrow x = \pm\sqrt{43}$.

  $$\left.\frac{dy}{dx}\right|_{(x,\, y)=(\pm\sqrt{43},\, 1)} = \frac{\pm 2\sqrt{43}}{7} \neq 0$$

  Therefore, the horizontal line $y = 1$ is not tangent to the curve.

  - A response that uses this method earns the first point by using $y = 1$ in $x^2 + 3y + 2y^2 = 48$.
  - A response that fails to consider both $x = +\sqrt{43}$ and $x = -\sqrt{43}$ does not earn the second point.

Total for part (b) 2 points

(c) The curve intersects the positive x-axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

At the point $(\sqrt{48}, 0)$, the slope of the line tangent to the curve is $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3+4(0)}$. The denominator of $\frac{dy}{dx}$ is $3+4(0)$, which does not equal 0. Therefore, the line tangent to the curve at this point is not vertical.	Answer with reason	**1 point**

Scoring notes:

- A response does not need to consider the numerator of $\left.\frac{dy}{dx}\right|_{(x,\, y)=(\sqrt{48},\, 0)}$ in order to earn this point; considering the denominator is sufficient.
- To earn this point a response must clearly demonstrate that the slope of the tangent line at the point $(\sqrt{48}, 0)$ is defined and answer "no."
  - Such demonstrations include, but are not limited to, the following:
    - $3 + 4(0) \neq 0$
    - At $(\sqrt{48}, 0)$, $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3}$.
    - $\frac{-2\sqrt{48}}{3}$
    - When $3 + 4y = 0$, $y \neq 0$.

Total for part (c) 1 point

(d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y-coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x-coordinate of the particle's position with respect to time?

$3y^2 \frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$	Attempts implicit differentiation	**1 point**
	$3y^2 \frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$	**1 point**
$\frac{dy}{dt} = -2$	Uses $\frac{dy}{dt} = \pm 2$	**1 point**
$3(2)^2(-2) + 2(4)(-2) + 2(2)\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{40}{4} = 10$ The rate of change with respect to time in the x-coordinate is 10 units per second.	Answer	**1 point**

Scoring notes:

- The first point is earned for implicitly differentiating $y^3 + 2xy = 24$ with respect to t with at most one error.
  - The first point can also be earned by correctly differentiating with respect to x: $3y^2 \frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$.
- The second point is earned for an equation equivalent to $3y^2 \frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$.
- A response does not need to explicitly declare $\frac{dy}{dt} = -2$ or $\frac{dy}{dt} = 2$ in order to earn the third point; this point may be earned by correctly substituting $\frac{dy}{dt} = -2$ or $\frac{dy}{dt} = 2$ in the implicitly differentiated equation. However, a response that uses both $\frac{dy}{dt} = -2$ and $\frac{dy}{dt} = 2$ in the implicitly differentiated equation does not earn the third point.
- The fourth point cannot be earned without the first 3 points. The fourth point is earned only for the value of 10 with no mistakes in supporting work.
  - Note that a response that uses $\frac{dy}{dt} = 2$ and then mishandles subtracting 40 from both sides of the equation, e.g., $3(2)^2(2) + 2(4)(2) + 2(2)\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{40}{4}$, does not earn the fourth point.

Total for part (d) **4 points**

Total for question 5 **9 points**

Part A (AB or BC): Graphing calculator required

Question 1 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

Model Solution	Scoring	

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t)\,dt$.

Model Solution	Scoring	
$\int_{60}^{135} f(t)\,dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds.	Interpretation with units	**1 point**
$\int_{60}^{135} f(t)\,dt$ $\approx f(90)(90-60) + f(120)(120-90) + f(135)(135-120)$ $= (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25$	Form of Riemann sum	**1 point**
	Answer	**1 point**

Scoring notes:

- To earn the first point the response must reference gallons of gasoline added/pumped and the time interval $t = 60$ to $t = 135$.
- To earn the second point at least five of the six factors in the Riemann sum must be correct.
- If there is any error in the Riemann sum, the response does not earn the third point.
- A response of $(0.15)(30) + (0.1)(30) + (0.05)(15)$ earns both the second and third points, unless there is a subsequent error in simplification, in which case the response would earn only the second point.

- A response that presents a correct value with accompanying work that shows the three products in the Riemann sum but does not show all six of the factors and/or the sum process, does not earn the second point but does earn the third point. For example, responses of either $4.5 + 3.0 + 0.75$ or $(0.15)(30), 0.1(30), 0.05(15) \rightarrow 8.25$ earn the third point but not the second.
- A response of $f(90)(90-60) + f(120)(120-90) + f(135)(135-120) = 8.25$ earns both the second and the third points.
- A response that presents an answer of only 8.25 does not earn either the second or third point.
- A response that provides a completely correct left Riemann sum with accompanying work, $f(60)(30) + f(90)(30) + f(120)(15) = 9$, or $(0.1)(30) + (0.15)(30) + (0.1)(15)$ earns 1 of the last 2 points. A response with any errors or missing factors in a left Riemann sum earns neither of the last 2 points.

Total for part (a) 3 points

(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

f is differentiable. $\Rightarrow$ f is continuous on $[60, 120]$.	$f(120) - f(60) = 0$	**1 point**
$\frac{f(120)-f(60)}{120-60} = \frac{0.1-0.1}{60} = 0$ By the Mean Value Theorem, there must exist a c, for $60 < c < 120$, such that $f'(c) = 0$.	Answer with justification	**1 point**

Scoring notes:

- To earn the first point a response must present either $f(120) - f(60) = 0$, $0.1 - 0.1 = 0$ (perhaps as the numerator of a quotient), or $f(60) = f(120)$.
- To earn the second point a response must:
  - have earned the first point,
  - state that f is continuous because f is differentiable (or equivalent), and
  - answer "yes" in some way.
- A response may reference either the Mean Value Theorem or Rolle's Theorem.
- A response that references the Intermediate Value Theorem cannot earn the second point.

Total for part (b) 2 points

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$. Show the setup for your calculations.

$\frac{1}{150-0}\int_0^{150} g(t)\,dt$	Average value formula	**1 point**
$= 0.0959967$	Answer	**1 point**

The average rate of flow of gasoline, in gallons per second, is 0.096 (or 0.095).	

Scoring notes:

- The exact value of $\frac{1}{150}\int_0^{150} g(t)\,dt$ is $\frac{12}{125}\sin\left(\frac{25}{16}\right)$.
- A response may present the average value formula in single or multiple steps. For example, the following response earns both points: $\int_0^{150} g(t)\,dt = 14.399504$ so the average rate is 0.0959967.
- A response that presents the average value formula in multiple steps but provides incorrect or incomplete communication (e.g., $\int_0^{150} g(t)\,dt = \frac{14.399504}{150} = 0.0959967$) earns 1 out of 2 points.
- A response of $\int_0^{150} g(t)\,dt = 0.0959967$ does not earn either point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{150}\int_0^{150} g(t)\,dt = 0.149981$ or 0.002618.

Total for part (c) 2 points

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

$g'(140) \approx -0.004908$ $g'(140) = -0.005$ (or -0.004)	$g'(140)$	**1 point**
The rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 (or 0.004) gallon per second per second at time $t = 140$ seconds.	Interpretation	**1 point**

Scoring notes:

- The exact value of $g'(140)$ is $\frac{1}{500}\cos\left(\frac{49}{36}\right) - \frac{49}{9000}\sin\left(\frac{49}{36}\right)$.
- The value of $g'(140)$ may appear only in the interpretation.
- To be eligible for the second point a response must present some numerical value for $g'(140)$.
- To earn the second point the interpretation must include "the rate of flow of gasoline is changing at a rate of [the declared value of $g'(140)$]" and "at $t = 140$" (or equivalent).
- An interpretation of "decreasing at a rate of -0.005" or "increasing at a rate of 0.005" does not earn the second point.
- Degree mode: In degree mode, $g'(140) = 0.001997$ or 0.00187.

Total for part (d) 2 points

Total for question 1 9 points