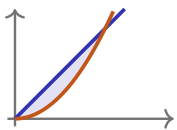
Applications of Integration

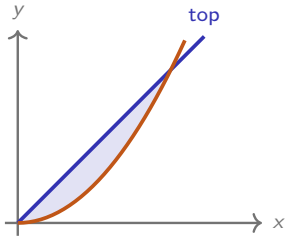
AP Calculus AB ▪ Unit 8

AP Calculus AB



What we will cover

1. Average value of a function
2. Motion with integrals
3. Net change in context
4. Area between curves
5. Volumes by cross section
6. Volumes of revolution



1 Average value

Average value of a function

The **average value** 平均値 over $[a, b]$:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

It is the constant height of a rectangle on $[a, b]$ with the same area as under f.

Do not confuse it with the average rate of change (which divides the *change in f* by the interval).

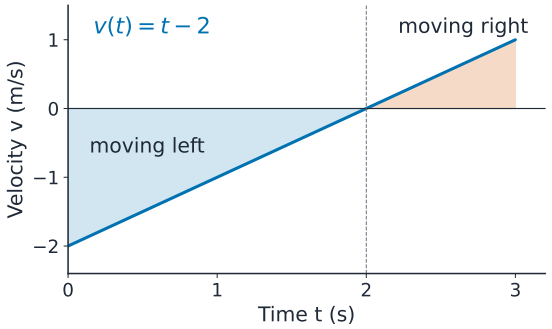
Average value of $f(x) = x^2$ on $[0, 3]$

$$\frac{1}{3} \int_0^3 x^2 dx = \frac{1}{3} \left[\frac{x^3}{3} \right]_0^3 = 3.$$

2

Motion

Read velocity before integrating



Your turn: teacher-created model

For $0 \leq t \leq 3$, find when direction changes. Predict whether displacement and total distance are equal. Explain using the graph.

Direction comes from the sign of velocity

$$v(t) = t - 2 = 0 \quad \Rightarrow \quad t = 2 \text{ s}.$$

Interval	$[0, 2)$	$(2, 3]$
Sign of v	$-$	$+$
Direction	left	right

Choose the quantity

Displacement 位移 counts signed change in position.

Total distance 总路程 counts both journeys positively.

Split at $t = 2$: this zero is accompanied by a sign change.

Evaluate the two journeys separately

$$\Delta x = \int_0^3 v(t)\,dt = \left[\frac{t^2}{2} - 2t\right]_0^3 = -1.5 \text{ m}.$$

$$D = \int_0^3 |v(t)|\,dt = -\int_0^2 (t-2)\,dt + \int_2^3 (t-2)\,dt.$$

$$D = |-2| + \frac{1}{2} = 2.5 \text{ m}.$$

Interpret the result

The particle travels 2 m left, then 0.5 m right. It ends 1.5 m left of its start, after travelling 2.5 m.

$\left|\int v\,dt\right|$ gives the magnitude of displacement; it misses the return journey.

Position needs the starting position

Your turn

The same particle starts at $x(0) = 5$ m. Find $x(3)$. Is $x(3)$ its displacement?

$$x(b) = x(a) + \int_a^b v(t)\,dt$$

The integral changes the starting position; it does not replace it.

Add signed change to the initial position

Recover position

$$x(3) = x(0) + \int_0^3 v(t)\,dt.$$

$$x(3) = 5 + (-1.5) = 3.5 \text{ m}.$$

Quantity	Value	Meaning
Final position	3.5 m	right of the origin
Displacement	-1.5 m	left of the starting point
Distance	2.5 m	length of the whole journey

Recover velocity from acceleration

Known quantities

$a(t) = 4\ \mathrm{m/s^2}$ and $v(0) = 1\ \mathrm{m/s}$. Find $v(t)$.

$$v(t) = v(0) + \int_0^t a(u)\,du$$

$$v(t) = 1 + \int_0^t 4\,du = 1 + 4t \quad \text{in metres per second.}$$

Equivalently, $v(t) = 4t + C$ and $v(0) = 1$ fixes $C = 1$. Each integration needs its own initial condition.

Does every zero mean a change of direction?

Test a claim

For $v(t) = (t-1)^2$, on $0 \le t \le 2$, a student says: "The particle turns at $t = 1$ because $v(1) = 0$." Decide whether the reason is sufficient.

A stop can occur without a turn

Check both sides

$v(t) > 0$ on both $(0, 1)$ and $(1, 2)$. The particle is momentarily at rest, but continues in the same direction.

Find zeros, then test the sign on each interval. Do not assume every root is a turning point.

Original exam: recover particle J position

2025 AB Q5D: calculator not allowed

Particle J has $v_J(t) = 2t(t^2 - 1)^3$ and $x_J(0) = 7$. Find $x_J(2)$, showing your work. The original gives no physical units.

1. Write an initial-position-plus-integral equation.
2. Identify a substitution from the derivative of $t^2 - 1$.

Original exam: set up before evaluating

$$x_J(2) = x_J(0) + \int_0^2 v_J(t)\,dt$$

$$x_J(2) = 7 + \int_0^2 2t(t^2-1)^3\,dt.$$

Substitution

$u = t^2 - 1$, so $du = 2t\,dt$.

$t = 0 \Rightarrow u = -1$; $t = 2 \Rightarrow u = 3$.

Keep 7 outside the integral. Velocity integrates to signed change.

Original exam: evaluate and interpret

$$x_J(2) = 7 + \int_{-1}^{3} u^3 \, du = 7 + \left[\frac{u^4}{4}\right]_{-1}^{3}$$
$$= 7 + \frac{81 - 1}{4}$$

$$x_J(2) = 27$$

The change in position is 20; the final position is 27. The official three points distinguish integrand, antiderivative and answer.

Original exam: position from a velocity table

2014 AB Q4(c)

Train A is 300 m east of the origin at $t = 2$ min. Use the three table intervals to estimate its position at $t = 12$ min by trapezoids.

t (min)	2	5	8	12
$v_A(t)$ (m/min)	100	40	-120	-150

Write the exact integral expression first. Preserve the signs and unequal interval widths.

Original exam: add the trapezoidal net change

$$s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt.$$

$$\Delta s \approx 3\frac{100+40}{2} + 3\frac{40-120}{2} + 4\frac{-120-150}{2}.$$

$$\Delta s \approx 210 - 120 - 540 = -450 \text{ m}.$$

$$s_A(12) \approx 300 - 450 = -150 \text{ m}.$$

About 150 m west of the origin. Table samples alone do not locate every zero, so taking absolute trapezoids is not an exact distance calculation.

Net change in an applied context

The **net change** 净变化 theorem is the everyday tool:

$$\text{final} = \text{initial} + \int_a^b (\text{rate})\, dt.$$

"How much water is in the tank at $t = 5$" $= \text{start} + \int_0^5 (\text{in} - \text{out})\, dt.$

> Watch the signs (in minus out) and the units.

Many multi-part FRQs are built entirely on this idea – often paired with "write, but do not evaluate," an integral expression.

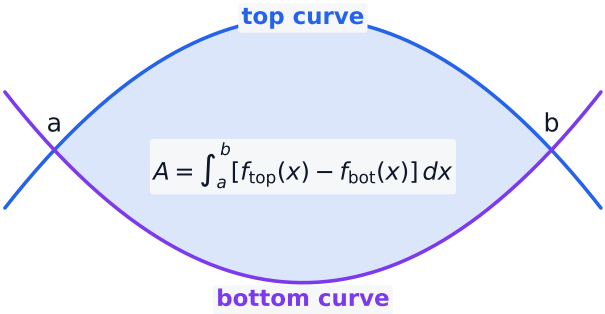
3 Area between curves

Area between two curves

On $[a, b]$ with f on top and g below:

$$A = \int_a^b \big[f(x) - g(x)\big]\,dx$$

Find the intersections first to set the limits, and always subtract bottom from top.



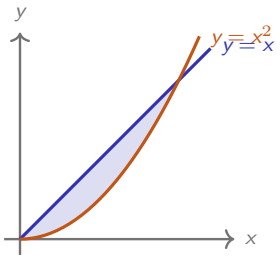
Worked example – the area between $y = x$ and $y = x^2$

Find the enclosed area

They cross at $x = 0$ and $x = 1$; on $(0, 1)$ the line is on top:

$$A = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$



Horizontal slices, and more than two crossings

Slice in y

$$A = \int_c^d \left[x_{\text{right}}(y) - x_{\text{left}}(y)\right] dy$$

Choose x- or y-slices to keep one top/bottom pair over the whole region.

Several crossings

$$A = \int_a^b \left|f(x) - g(x)\right| dx$$

or split into a sum at each crossing – decide which curve is higher on each piece.

4 Volumes

Volumes with known cross sections

If a solid has a known **cross section** 横截面 perpendicular to an axis, the volume is the integral of its area:

$$V = \int_a^b A(x)\,dx$$

The exam phrase "the base of a solid… cross sections are squares" signals exactly this.

square of side s: $A = s^2$

rectangle, height ks: $A = k\,s^2$

equilateral **triangle**: $A = \frac{\sqrt{3}}{4}s^2$

semicircle, diameter s: $A = \frac{\pi}{8}s^2$

Set the shape's key length equal to the slice length $s(x) = f(x) - g(x)$, then integrate.

Volumes of revolution – the disc method

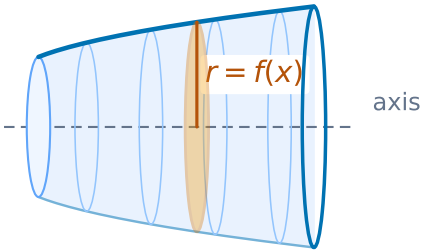
Revolving a region about an axis makes a **solid of revolution** 旋转体. If the region touches the axis, each slice is a **disc** 圆盘:

$$V = \pi \int_a^b \left[r(x)\right]^2 dx$$

About the y-axis: express x as a function of y and integrate in y.

Volumes of revolution – the disc method, in detail

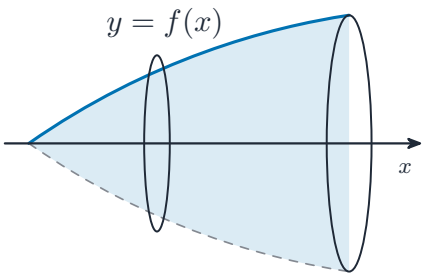
rotate $y = f(x)$ about the x-axis



$$V = \pi \int_a^b [f(x)]^2 \, dx \quad \text{(disk method about the } x\text{-axis)}$$

Revolving a region about an axis makes a **solid of revolution** 旋转体. Where the region touches the axis, every slice is a **disc** 圆盘.

Volume with the Disc Method (About the x- or y-Axis)



rotate the region around the x-axis to sweep a solid

Rotating a region about an axis sweeps out a solid of revolution

Volumes with Cross Sections: Squares and Rectangles

A potter shaping a vessel: solids of revolution are volumes formed by rotating a plane region

Exam tips

- Area between curves is $\int (\text{top} - \text{bottom})\, dx$ – find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f\, dx$.
- Accumulated change is $\int$ of a rate: total $=$ initial value $+ \int_a^b (\text{rate})\, dt$.
- Integration means "adding up infinitely many tiny pieces" – set up the integrand as one thin slice.

5 Recap

Unit 8 – key takeaways

1 Average value

$\frac{1}{b-a}\int_a^b f$ – not the average rate of change

2 Motion

displacement integrates v; distance integrates $|v|$

3 Net change

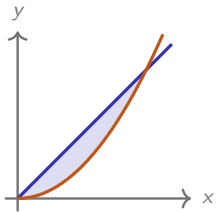
final = initial + $\int$ rate – watch signs and units

4 Area & volume

top minus bottom; $V = \int A(x)\,dx$; discs use πr^2

Check yourself

1. Which integral gives total distance, $\int v$ or $\int |v|$?
2. Find the area between $y = x$ and $y = x^2$.
3. Write the average value of f on $[a, b]$.
4. A solid's cross sections are squares of side $s(x)$. Give V.



Answers **1.** $\int |v|$ **2.** $\frac{1}{6}$ **3.** $\frac{1}{b-a}\int_a^b f$ **4.** $V = \int_a^b [s(x)]^2 dx$