Differential Equations

AP Calculus AB ▪ Unit 7

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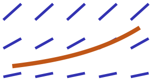


What we will cover

1. Modelling with differential equations
2. Verifying a solution
3. Slope fields
4. Separation of variables
5. Particular solutions
6. Exponential growth & decay

$$\frac{dy}{dt} = ky$$

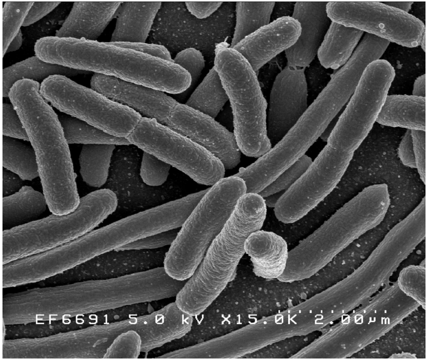
$$\downarrow$$

$$y = y_0 e^{kt}$$

Modelling a situation with a rate

A **differential equation** 微分方程 relates a function to its own derivatives – it describes a situation by its rate of change. "The rate of change of a quantity is proportional to its size" becomes

$$\frac{dy}{dt} = ky.$$



bacteria Growth

Verifying a proposed solution

A **solution** 解 is a function that makes the equation true. To verify one: differentiate it and substitute – if both sides match, it is a solution.

A differential equation usually has infinitely many solutions – a whole family, differing by a constant.

Check $y = 2e^{x^2/2}$ solves $\frac{dy}{dx} = xy$

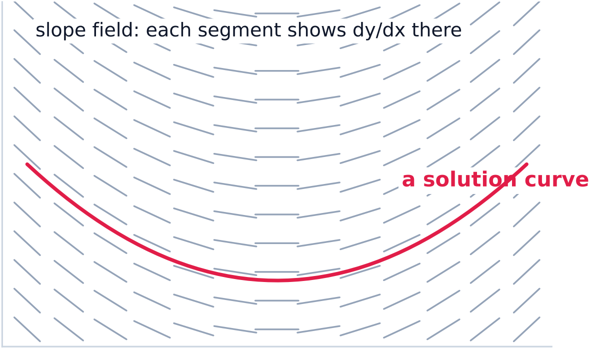
$$\frac{dy}{dx} = 2e^{x^2/2} \cdot x = x\left(2e^{x^2/2}\right) = xy.$$

True.

Slope fields

A **slope field** 斜率场 draws the equation as a grid of short segments: at each (x, y) the segment has slope $\dfrac{dy}{dx}$ there.

> It shows the shape of the solution curves without solving.



Reading a slope field

segments **flat** $\Rightarrow \frac{dy}{dx} = 0$: the solution is momentarily level

segments **steepen** $\Rightarrow$ the solution rises or falls faster

a **horizontal row** of equal slopes $\Rightarrow$ the rate depends only on y

Exam skill: start at the given point and follow the segments

A solution curve stays tangent to the field everywhere.

Separation of variables

If $\dfrac{dy}{dx}$ factors into a function of x times a function of y, split them and integrate:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

> Add the constant of integration once – on the x-side.

all y one side, all x the other

↓

integrate both sides

↓

the **general solution** 通解

Worked example – separate, integrate, then pin it down

Solve $\frac{dy}{dx} = xy$ with $y(0) = 2$

$$\int \frac{dy}{y} = \int x\,dx \;\Rightarrow\; \ln|y| = \frac{x^2}{2} + C$$

$$\Rightarrow\; y = Ae^{x^2/2}$$

Applying $y(0) = 2$ gives $A = 2$:

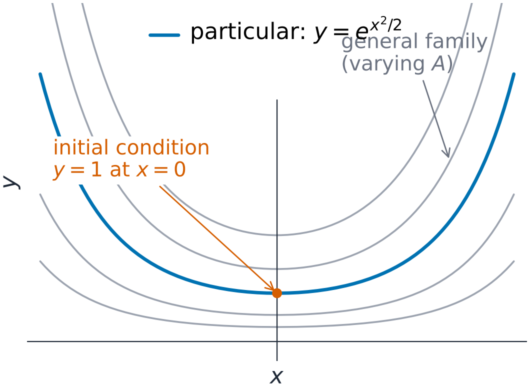
$$y = 2e^{x^2/2}.$$

An **initial condition** 初始条件 (x_0, y_0) picks one curve from the family – there is exactly one solution through a given point.

From the family to the particular solution

The constant gives a family of curves; the initial condition selects the **particular solution** 特解.

> Watch **domain restrictions** 定义域限制: keep the branch containing the initial point (e.g. the right sign of a square root).



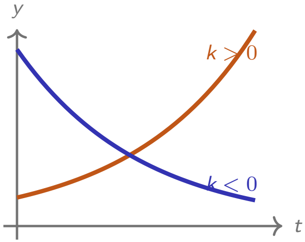
Exponential growth and decay

The most important model: rate proportional to size.

$$\frac{dy}{dt} = ky \implies y = y_0 e^{kt}$$

where y_0 is the value at $t = 0$.

> $k > 0$ gives **growth** 增长; $k < 0$ gives **decay** 衰减.



Worked example – radioactive decay and half-life

$\frac{dy}{dt} = -0.10\,y$ (years), $y_0 = 50$ g

The solution is $y = 50e^{-0.10t}$, so after 10 years

$$y = 50e^{-1} = 18.4 \text{ g}.$$

Half-life 半衰期: solve $25 = 50e^{-0.10t}$, i.e. $e^{-0.10t} = \frac{1}{2}$:

$$t = \frac{\ln 2}{0.10} = 6.9 \text{ years}.$$

The half-life is independent of the starting amount – that is what makes it a useful constant.

Unit 7 – key takeaways

1 Modelling

a sentence about a rate becomes an equation in dy/dt

2 Slope fields

segments show the slope; a solution follows them

3 Separation

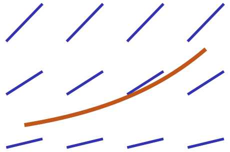
split the variables, integrate, add $+C$ once

4 Exponentials

$dy/dt = ky \Rightarrow y = y_0 e^{kt}$ – growth or decay

Check yourself

1. Write “the rate of decay is proportional to the amount”.
2. Solve $\frac{dy}{dx} = xy$ in general form.
3. How many times do you add $+C$ when separating?
4. Does half-life depend on the starting amount?



a solution follows the field

Answers **1.** $\frac{dy}{dt} = -ky$ **2.** $y = Ae^{x^2/2}$ **3.** once, on the x-side **4.** no