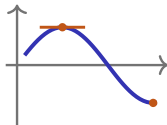


Analytical Applications of Differentiation

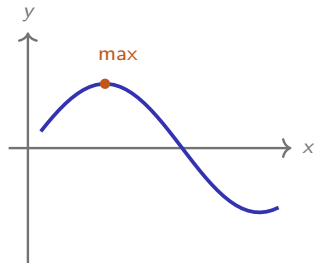
AP Calculus AB ■ Unit 5

AP Calculus AB



What we will cover

- 1 The Mean Value Theorem
- 2 Critical points & extrema
- 3 The first derivative test
- 4 Concavity & inflection
- 5 Connecting f , f' and f''
- 6 Optimization



1

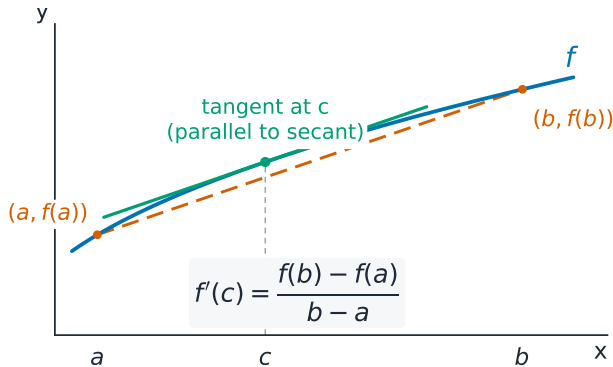
Mean Value Theorem

The Mean Value Theorem

If f is **continuous** on $[a, b]$ and **differentiable** on (a, b) , then for some c in (a, b) :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Somewhere inside, the **instantaneous** rate equals the **average** rate – some tangent is **parallel to the secant**.



Exam skill – justifying the MVT

The **Mean Value Theorem** 中值定理 is an **existence** theorem, so the marks are in the justification:

- 1 state f is continuous on $[a, b]$ and differentiable on (a, b)
- 2 compute $\frac{f(b) - f(a)}{b - a}$
- 3 conclude “by the MVT there is a c in (a, b) with $f'(c)$ equal to that value”

Both hypotheses must be named – dropping one loses the mark even with the right c .

Verify the MVT for $f(x) = x^2$ on $[1, 3]$

A polynomial, so continuous and differentiable everywhere. Average rate $\frac{9 - 1}{2} = 4$; solve $f'(c) = 2c = 4$ to get $c = 2$, which lies in $(1, 3)$.

2

Critical points

Extreme values and critical points

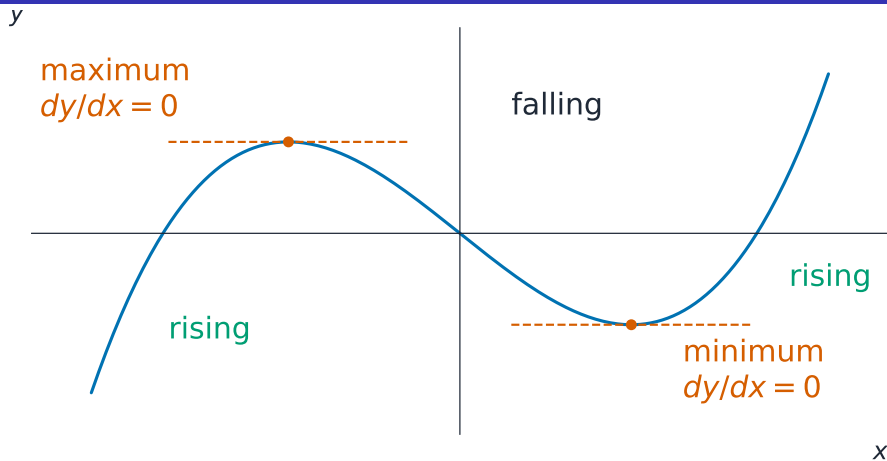
The **Extreme Value**

Theorem 极值定理: a function **continuous** on a **closed** interval attains **both** an absolute max and an absolute min. A **critical point** 临界点 is an interior point where $f'(x) = 0$ **or** f' does not exist.



Fixed fencing, largest area: an optimisation problem

At a differentiable interior extremum the derivative is zero

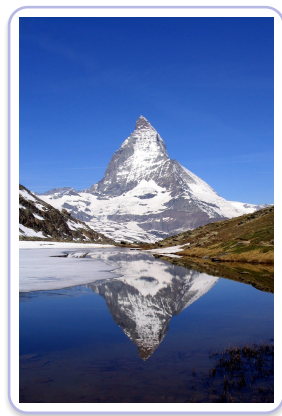


If f has a local extremum at c and $f'(c)$ exists, then $f'(c) = 0$. The converse fails: $f(x) = x^3$ has $f'(0) = 0$ but no extremum.

Increasing and decreasing intervals

The **first derivative** says where f rises or falls:

- $f'(x) > 0 \Rightarrow f$ is **increasing** 递增
- $f'(x) < 0 \Rightarrow f$ is **decreasing** 递减



*Uphill, summit, downhill: $f' > 0$, $f' = 0$,
 $f' < 0$*

The first derivative test

Classify a critical point c by how f' **changes sign** there:

f' turns $+$ $\rightarrow$ $-$

local maximum

极大值

f' turns $-$ $\rightarrow$ $+$

local minimum

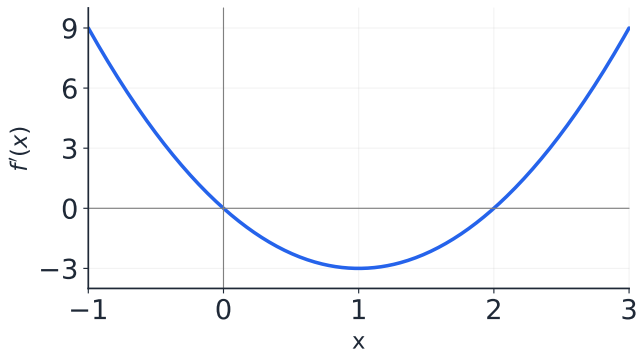
极小值

f' does **not** change sign

neither

Always state the **sign change** as your justification.

Read a derivative graph to describe the function



Your turn

The graph is $f'(x) = 3x^2 - 6x$ on $(-1, 3)$. Where is f increasing or decreasing? Classify the critical points and justify each answer.

The vertical coordinate is f' , not f . Use the sign on each interval.

Use signs, then name the turning points

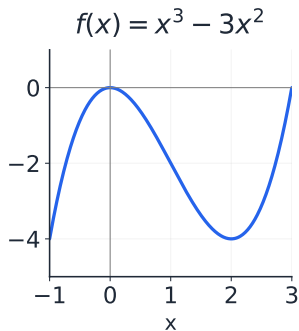
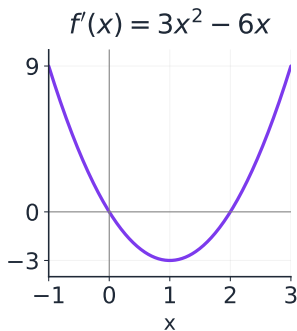
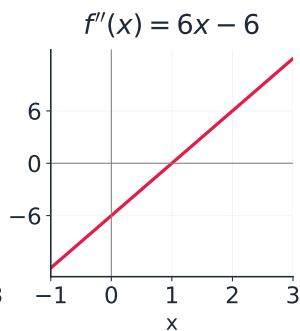
$$f'(x) = 3x(x - 2) = 0 \implies x = 0, 2.$$

Interval	$(-1, 0)$	$(0, 2)$	$(2, 3)$
Sign of f'	+	-	+
Behaviour of f	increasing	decreasing	increasing

State the evidence

At 0, f' changes from positive to negative: a local maximum. At 2, it changes from negative to positive: a local minimum. Zeros alone would only identify candidates.

Match all three graphs to the same cubic



For $f(x) = x^3 - 3x^2$, the extrema are $(0, 0)$ and $(2, -4)$. The derivative zeros align with these x coordinates; the graphs do not have the same heights.

A derivative graph does not fix the vertical position

$$f(x) = x^3 - 3x^2, \quad F(x) = x^3 - 3x^2 + 5.$$

$$F'(x) = f'(x) = 3x^2 - 6x.$$

What stays the same?

Both functions increase and decrease on the same intervals, and their turning points have the same x coordinates. Their y coordinates differ by 5. A value such as $f(0) = 0$ fixes which vertical position is intended.

Sketch the shape from f' ; use a given function value to place it vertically.

The candidates test – absolute extrema

On a **closed** interval, absolute extrema occur only at **critical points or endpoints**:

- 1 list every critical point in $[a, b]$ **and** both endpoints
- 2 evaluate f at each candidate
- 3 largest is the absolute max; smallest the absolute min

Show the **table of values** – the comparison *is* the argument.

endpoints a, b
+
every critical point
= the full candidate list

Extrema: local, global, and how to justify

Local (relative) extrema

Local (relative) extrema are the highest or lowest value in some neighbourhood – and by Fermat's theorem they occur only at critical points.

Absolute extrema

On a closed interval these are found by the candidates test: evaluate at every critical point *and* both endpoints.

Justifying

The first-derivative test names the sign change of f' ; the second-derivative test uses the sign of f'' at the critical point.

What earns the mark

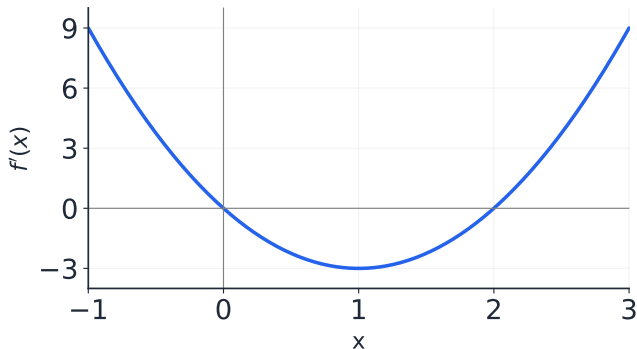
State the value *and* the reason – “ f' changes from positive to negative at $x = 2$, so $f(2)$ is a local maximum”.

A critical point is a candidate, never a conclusion. The justification is the marked part of an AP answer.

3

Concavity

Above the axis, but sloping down



Your turn

For $f(x) = x^3 - 3x^2$, decide whether f increases on $(-1, 0)$ and whether it is concave up there. Locate its inflection; justify using the graph of f' .

Height of f' answers one question; increasing or decreasing f' answers another.

Positive derivative can accompany concave down

$$f'(x) = 3x^2 - 6x, \quad f''(x) = 6x - 6.$$

On $(-1, 0)$

$f' > 0$, so f increases. But f' decreases, so f is **concave down** 下凹. These conclusions are compatible.

Across $x = 1$

$f'' < 0$ for $x < 1$ and $f'' > 0$ for $x > 1$. Concavity changes from down to **concave up** 上凹, giving a **point of inflection** 拐点 at $(1, -2)$ for the stated cubic. Here $f'(1) = -3$, not zero.

Does a zero second derivative prove inflection?

$$h(x) = x^4, \quad k(x) = x^3$$

Your turn, at $x = 0$

Both second derivatives are zero. Find their second derivatives and test the signs on both sides. Which graph has an inflection point? Does either function have a local minimum?

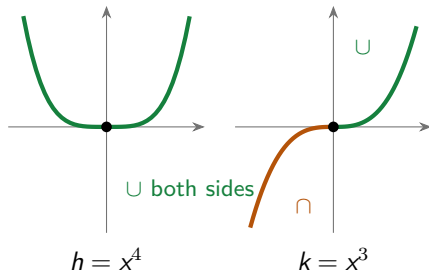
A candidate needs a sign or concavity check.

The same candidate gives different conclusions

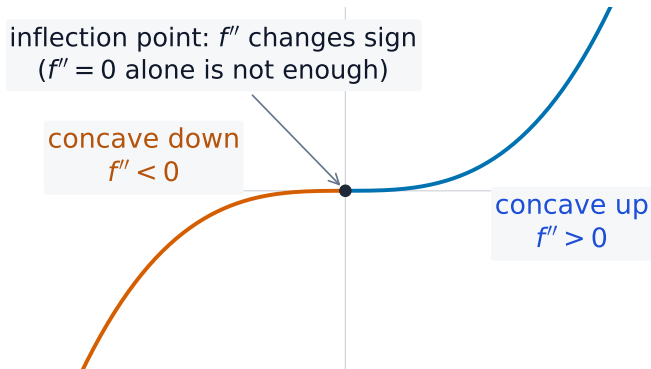
	$h(x) = x^4$	$k(x) = x^3$
$f'(x)$	$12x^2$	$6x$
signs across 0	$+ \rightarrow +$	$- \rightarrow +$
inflection at 0?	no	yes

$f'(c) = 0$ is only a **candidate**: an inflectionion needs f' to **change sign**.

h even has a local **minimum** at 0 although $h''(0) = 0$.



Concavity and points of inflection, in detail



Separate example $f(x) = x^3/6$: concavity changes at 0.

The second derivative test

An alternative at a critical point c with $f'(c) = 0$:

$$f''(c) > 0 \Rightarrow \text{local minimum}$$

$$f''(c) < 0 \Rightarrow \text{local maximum}$$

$$f''(c) = 0 \Rightarrow \text{inconclusive}$$

If $f''(c) = 0$, fall back on the **first derivative test**.

Special case: if a continuous function has only **one** critical point on an interval and it is a local extremum, it is the **absolute** extremum there.

Connecting f , f' and f''

A very common setup: you are given the graph of f' and asked about f .

f' above the axis $\Rightarrow f$ **increasing**

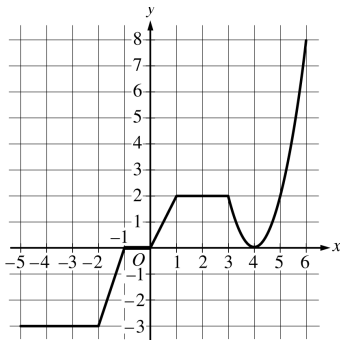
f' crosses $+$ $\rightarrow$ $-$
 $\Rightarrow f$ has a **local max**

f **concave up** $\Leftrightarrow f'$ **increasing**

At a continuous point, a change in increasing/decreasing behaviour of f' identifies a change in concavity of f

Answer questions about f using the **height** and **slope** of the f' graph.

Original 2018: the graph is g , and g equals f -prime



Graph of g

2018 AB Q 3(c–d)

Given $g = f'$, find intervals where f is both increasing and concave up. Then find the x coordinate of each inflection point of f . Justify using the given graph.

Original 2018: combine height and direction of change

$$f'(x) = g(x) > 0 \quad \text{and} \quad g \text{ increasing.}$$

Increasing and concave up

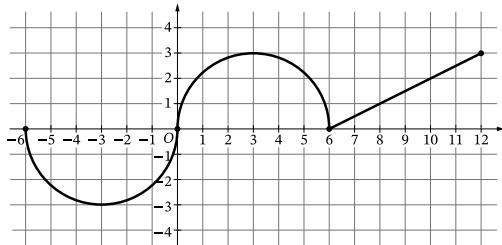
Both hold on $(0, 1)$ and $(4, 6)$. A positive but horizontal graph of g does not establish concave up.

Inflection

At 4, $g = f'$ changes from decreasing to increasing, so f changes concavity. Although $g(4) = 0$, g is positive on both sides: this point is not an extremum of f .

Give both intervals and the graph-based reason.

Original 2025: now the graph is f , and f equals g' -prime



Graph of f

2025 AB Q 4B

$g(x) = \int_6^x f(t) dt$, so $g' = f$. Find all inflection x coordinates of g on $(-6, 12)$. Is every zero of the displayed graph an inflection? Give graph-based reasons.

Original 2025: turns of f locate inflections of g

x	Change in $f = g'$
-3	decreasing to increasing
3	increasing to decreasing
6	decreasing to increasing

Why 0 is excluded, and 6 is included

At 0, f increases on both sides, so g stays concave up. At 6, f changes direction although f' is undefined. The neighbouring concavity changes; g'' need not exist at the point itself.

Justify each answer using the increasing/decreasing behaviour of the displayed f .

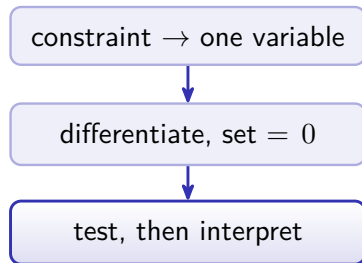
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Optimization

Solving an optimization problem

Optimization 最优化 is the same machinery aimed at a real goal:

- 1 write the quantity as a function of **one** variable (use the constraint to eliminate the rest)
- 2 state the allowed interval
- 3 find and **test** the critical points
- 4 answer with **units** and interpretation



Worked example – the farmer's fence

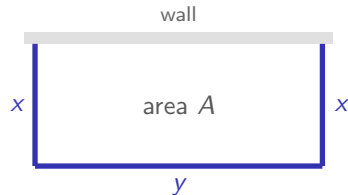
100 m of fence, three sides (a wall is the fourth). Maximize the area.

Ends x , side y : constraint $2x + y = 100$, so $y = 100 - 2x$.

$$A(x) = x(100 - 2x) = 100x - 2x^2$$

$$A'(x) = 100 - 4x = 0 \Rightarrow x = 25$$

$A'' = -4 < 0$, a maximum: $y = 50$ and $A = 1250 \text{ m}^2$.



Behaviour of implicit relations

It all extends to **implicitly** defined relations:

- horizontal tangent where $\frac{dy}{dx} = 0$
- vertical tangent where $\frac{dy}{dx}$ is **undefined**

$\frac{dy}{dx}$ usually involves **both** x and y – substitute it back in when finding $\frac{d^2y}{dx^2}$, then read concavity from the sign.

5

Recap

Unit 5 – key takeaways

1

MVT

some tangent is parallel to the secant – name both hypotheses

2

Critical points

$f' = 0$ or undefined: candidates only, so test each

3

The two tests

f' sign change, or $f'' > 0$ min / $f'' < 0$ max

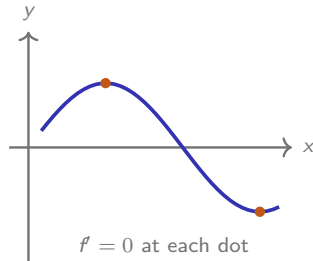
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Justify

cite the sign of f' or f'' – the reasoning earns the marks

Check yourself

- 1 f' turns $-$ to $+$ at c . What is c ?
- 2 $f'(c) = 0$ at a critical point. What does the test tell you?
- 3 Where can an absolute max on $[a, b]$ occur?
- 4 Is every critical point an extremum?



Answers 1. a local minimum 2. inconclusive – use the f' test 3. a critical point or an endpoint 4. no