Contextual Applications of Differentiation

AP Calculus AB ▪ Unit 4

AP Calculus AB

$s(t) \xrightarrow{d/dt} v(t) \xrightarrow{d/dt} a(t)$

What we will cover

$s(t) \rightarrow v(t) = s' \rightarrow a(t) = v'$

1 In context

Interpreting a derivative – units and a sentence

$f'(x)$ is the instantaneous rate of change of f per unit of x.

The **unit** 单位 of f' is unit of f ÷ unit of x: acres per week, litres per minute.

"Interpret $g'(140) = 2.3$ with units"

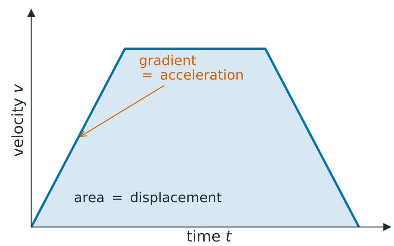
A full-credit sentence needs four parts: *At* $x = 140$ (the moment), *the quantity is increasing* (the direction) *at about* 2.3 *units* (the value) ***per*** *unit of* x (the rate word).

2 Motion

Straight-line motion

Three functions of time, linked by differentiation:

- **position** 位置 $s(t)$
- **velocity** 速度 $v(t) = s'(t)$ – signed
- **acceleration** 加速度 $a(t) = v'(t) = s''(t)$



Reading the motion – the frequent exam parts

At rest 静止
$v(t) = 0$

Direction
right/up if $v > 0$.
left/down if

Changes direction
where v changes sign

Speed
$|v(t)|$ – no direction

A rising balloon: related rates link how fast radius, volume and height change together

Speeding up or slowing down?

Compare the signs of v and a:

Same sign ⇒ speeding up. Opposite signs ⇒ slowing down.

Velocity has direction; speed does not. The exam tests exactly this difference.

Roller coaster

Worked example – a particle on a line

$s(t) = t^3 - 6t^2 + 9t$

$$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$$

At rest at $t = 1$ and $t = 3$; it changes direction at each. At $t = 2$: $v(2) = -3 < 0$ (moving left), and just after, $a > 0$ while $v < 0$ – opposite signs, so it is slowing down.

$s(t) = t^3 - 6t^2 + 9t$
↓
$v(t) = 3(t-1)(t-3)$
↓
$a(t) = 6t - 12$

3 Related rates

Related rates – the five-step template

In a **related rates** 相关变化率 problem, quantities change together in time. The engine is the chain rule: differentiate with respect to t.

1. name the variables; list given & wanted rates
2. write an equation relating them
3. differentiate both sides in t
4. substitute the values last
5. state units and the correct sign

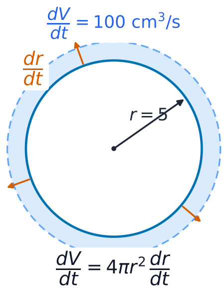
Worked example – the inflating balloon

$\frac{dV}{dt} = 100\ \mathrm{cm}^3/\mathrm{s}$: how fast is r growing at $r = 5$?

Start from $V = \frac{4}{3}\pi r^3$ and differentiate in t first:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$100 = 4\pi(5)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi} \approx 0.32\ \mathrm{cm/s}.$$



The classic related-rates error

Substitute $r = 5$ first
$V = \frac{4}{3}\pi(5)^3$ is a constant, so $\frac{dV}{dt} = 0$. The radius is now frozen – the problem is destroyed.

Differentiate first
Keep r a function of t, differentiate the general relationship, *then* put $r = 5$ in.

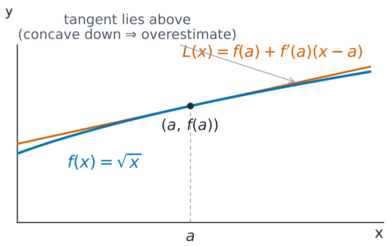
Substituting the instant's value too early is the single most common lost mark here.

4 Approximation

Local linearity and linearization

Near the point of tangency a smooth curve looks like its tangent – **local linearity** 局部线性. So the tangent is the best **linear approximation** 线性近似:

$f(x) \approx L(x) = f(a) + f'(a)(x - a)$.



Worked example – estimate $\sqrt{4.1}$

$f(x) = \sqrt{x}$, $a = 4$

$f(4) = 2$ and $f'(x) = \dfrac{1}{2\sqrt{x}}$, so $f'(4) = \frac{1}{4}$:

$$L(4.1) = 2 + \tfrac{1}{4}(4.1 - 4) = 2.025$$

(true value $2.0248\ldots$).

$\sqrt{x}$ is concave down ($f'' < 0$), so the tangent lies above the curve – a slight **overestimate** 高估.

Justify over/under with the sign of f'', never by eye.

5 L'Hospital

L'Hospital's rule

If substitution gives the **indeterminate form** 未定式 $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Differentiate top and bottom separately – this is not the quotient rule.

Evaluate $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

Substituting gives $\frac{0}{0}$, so differentiate top and bottom:

$$\lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

– the famous limit from Unit 1.

Check the form first – applying the rule to anything else is a mistake.

6 Recap

Unit 4 – key takeaways

1 In context

f' is a rate: units of f per unit of x – always interpret

2 Motion

$s \to v \to a$; same signs speed up, opposite slow down

3 Related rates

differentiate in t FIRST, substitute last

4 Approximation

tangent line estimates f nearby; f'' gives over/under

Check yourself

1. $v < 0$ and $a < 0$. Speeding up or slowing down?
2. In a related-rates problem, when do you substitute the numbers?
3. $C(t)$ is acres, t is weeks. Units of $C'(t)$?
4. Which form must you confirm before using L'Hospital?

$$s(t) \downarrow v(t) \downarrow a(t)$$

Answers **1.** speeding up (same sign) **2.** last, after differentiating **3.** acres per week **4.** $\frac{0}{0}$ or $\frac{\infty}{\infty}$