Limits & Continuity

AP Calculus AB ▪ Unit 1

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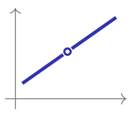


What we will cover

1 Change at an instant

Can change occur at an instant?

A speedometer reads 60 km/h. But at a single instant no time passes and no distance is covered:

$$\text{speed} = \frac{0}{0} \text{ – undefined.}$$

The clever move: do not plug in zero. Watch what the average rate **approaches** 趋近 as the interval shrinks.

That approaching value is a **limit** 极限 – it powers the derivative (Unit 2) and, in reverse, the integral (Unit 6).

2 Reading a limit

Defining a limit

$\lim_{x \to c} f(x) = R$ means $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c – but not equal to c.

A limit describes the behaviour near c, not the value at c.

graphically 用图象 **numerically** 用数值

analytically 用解析式

Moving between the three is a core skill.

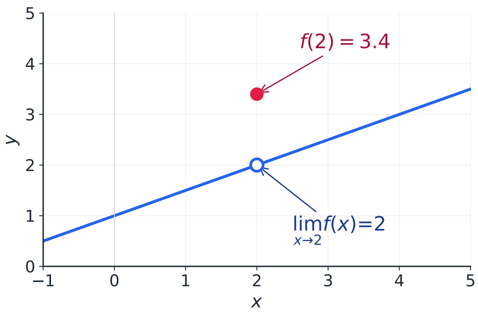
Reading a limit from a graph

Trace toward c from each side:

- from the left: **left-hand limit** 左极限
- from the right: **right-hand limit** 右极限

If both head to the same height, the two-sided limit exists.

> **Open circle** 空心圆: a height approached but not reached.
> **Closed circle** 实心圆: the actual value.



When a limit does not exist

Jump 跳跃	**Unbounded** 无界	**Oscillates** 振荡
the two sides disagree	grows without limit	forever near c
$\lim_{x\to 0} \frac{\lvert x\rvert}{x}$ DNE	$\lim_{x\to 0} \frac{1}{x^2} = \infty$	$\lim_{x\to 0} \sin\frac{1}{x}$ DNE

> Watch the **scale** 比例: a zoomed-out graph can hide behaviour near a point – confirm with algebra.

Estimating a limit from a table

Estimate $\lim_{x\to 2} \frac{x^2-4}{x-2}$ (it is $\frac{0}{0}$ at $x = 2$)

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4.

> A table only suggests a value – it is a numerical estimate, not a proof.

3 Computing limits

The limit theorems

If both limits exist, the limit of a combination is that combination of the limits:

$$\lim(f \pm g) = \lim f \pm \lim g$$

$$\lim(fg) = \lim f \cdot \lim g$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad (\text{bottom} \neq 0)$$

$$\lim g(f(x)) = g(\lim f(x)) \quad (g \text{ continuous})$$

The practical rule: try **direct substitution** 直接代入 first. A real number out means you are done.

When substitution gives $\frac{0}{0}$

$\frac{0}{0}$ is an **indeterminate form** 未定式 – it does not mean the limit fails. Rewrite into an **equivalent form** 等价形式, then substitute.

The cancelled factor is why the graph had a hole: the two functions agree everywhere except at c.

Three standard moves

Factor and cancel 因式分解并约分:

$$\lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$

Multiply by the **conjugate** 共轭:

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{1}{2}$$

Or use a trig identity.

Choosing a procedure

1. substitute – a real answer means done

2. $\frac{0}{0}$? rewrite, then substitute

3. $\frac{5}{0}$? infinite or DNE – check each side

4. $x \to \pm\infty$? compare the **dominant** 主导 terms

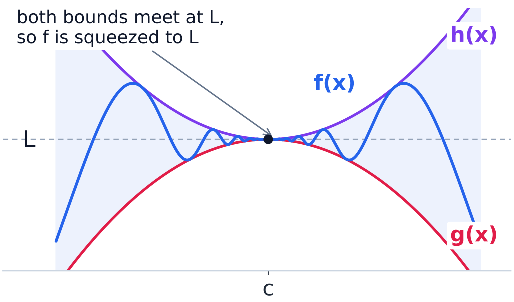
5. trapped between two functions? squeeze theorem

The squeeze theorem

If $g(x) \leq f(x) \leq h(x)$ near c and both bounds meet at L, then f has nowhere to go:

$$\lim_{x \to c} f(x) = L.$$

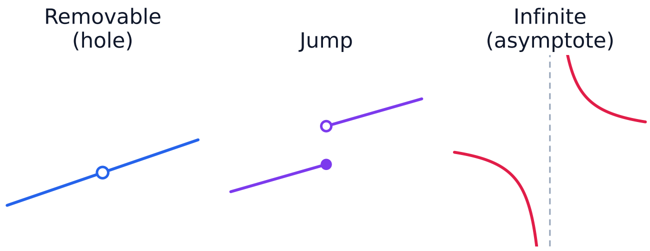
The two famous results proved this way: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$



4 Continuity

Three types of discontinuity

A function is **discontinuous** 间断 at c when its graph breaks there:



Removable 可去间断 (a hole) ▪ **jump** 跳跃间断 (sides disagree) ▪ **infinite** 无穷间断 (an asymptote).

Continuity at a point – the three-part test

f is **continuous** 连续 at $x = c$ exactly when all three hold:

- $f(c)$ **exists**
- $\lim_{x \to c} f(x)$ **exists**
- they **agree**: $\lim_{x \to c} f(x) = f(c)$

The point is there, the limit is there, and the two agree. If any one fails, f is discontinuous.

Learn it as a checklist – it is the backbone of nearly every continuity question.

Continuity over an interval, and repairing a hole

You rarely check point by point – whole families are continuous on their domains:

Polynomial, rational, power, **exponential** 指数, **logarithmic** 对数 and **trigonometric** 三角 functions are continuous at every point of their domains.

So a rational function is continuous everywhere except where its denominator is zero.

Removing a discontinuity

If the limit exists at a hole, redefine the function there to equal the limit – the graph is repaired.

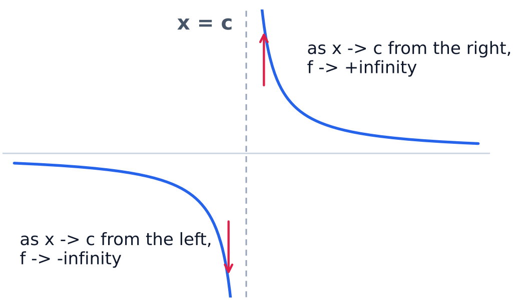
For a **piecewise** 分段函数 function at a boundary, solve for the **parameter** 参数 that makes the pieces meet: $\lim_{x \to c^-} f = \lim_{x \to c^+} f = f(c)$.

5 Asymptotes

Infinite limits and vertical asymptotes

Where a non-zero number is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** 垂直渐近线.

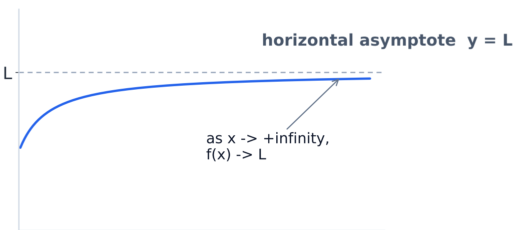
> Always check each side: the two sides can shoot opposite ways.



Limits at infinity and horizontal asymptotes

Let the input grow: **end behaviour** 末端行为 as $x \to \pm\infty$. If outputs settle toward L, then $y = L$ is a **horizontal asymptote** 水平渐近线.

> Far out, an exponential beats any polynomial, and a polynomial beats any logarithm.



Rational functions – compare the degrees

top < bottom limit is 0 (asymptote $y = 0$)	top = bottom the **ratio of leading coefficients** 首项系数之比	top > bottom unbounded – no horizontal asymptote

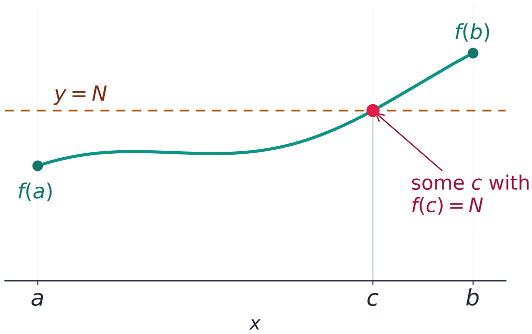
> "As $t \to \infty$, where does the rate settle?" is answered with a limit at infinity.

6 IVT

The Intermediate Value Theorem

If f is continuous on $[a, b]$ and d lies between $f(a)$ and $f(b)$, then $f(c) = d$ for at least one c in (a, b).

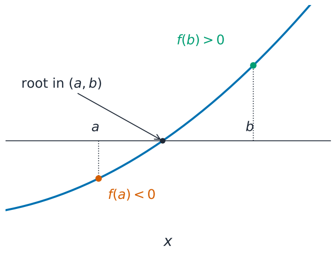
> An unbroken curve cannot skip a height between its endpoints.



Exam skill – justifying with the IVT

These appear almost every year. A full-credit justification has three moves:

1. state continuity on $[a, b]$
2. show d is trapped: $f(a) < d < f(b)$
3. conclude by name: "by the IVT there is a c in (a, b) with $f(c) = d$"



Skipping the continuity statement – or not showing d is between the endpoints – loses the point.

Worked example – a hole at $x = 3$

Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Direct substitution gives $\frac{0}{0}$, so factor:

$$\frac{(x-3)(x+3)}{x-3} = x + 3 \quad (x \neq 3)$$

$$\lim_{x \to 3}(x + 3) = 6.$$

The graph has a removable discontinuity at $x = 3$ – the limit exists even though the function is undefined there.

7 Recap

Unit 1 – key takeaways

1 A limit

what f approaches near c – not the value at c

2 Computing

substitute first; on $0/0$ factor, cancel or rationalise

3 Continuity

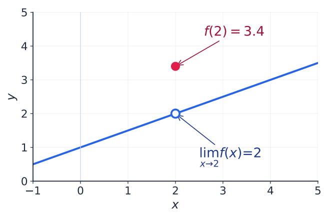
value exists, limit exists, and the two agree

4 The IVT

continuous $+$ d trapped $\Rightarrow$ some c hits d

Check yourself

1. Evaluate $\lim_{x\to 3} \frac{x^2-9}{x-3}$.
2. Must $\lim_{x\to c} f(x)$ equal $f(c)$?
3. Name the three parts of the continuity test.
4. Which two conditions must an IVT justification state?



Answers **1.** 6 **2.** no – only if f is continuous at c **3.** $f(c)$ exists, the limit exists, they agree **4.** continuity, and d between $f(a)$ and $f(b)$