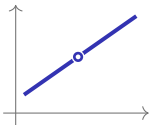


Limits & Continuity

AP Calculus AB ■ Unit 1

AP Calculus AB



What we will cover

- 1 Change at an instant
- 2 Reading a limit
- 3 Computing limits
- 4 The squeeze theorem
- 5 Discontinuity & continuity
- 6 Asymptotes
- 7 The Intermediate Value Theorem

1

Change at an instant

Can change occur at an instant?

A speedometer reads 60 km/h. But **at a single instant** no time passes and no distance is covered:

$$\text{speed} = \frac{0}{0} - \text{undefined.}$$

The clever move: do **not** plug in zero. Watch what the average rate **approaches** 趋近 as the interval shrinks.

That approaching value is a **limit** 极限 – it powers the **derivative** (Unit 2) and, in reverse, the **integral** (Unit 6).

What calculus is for, and the words limits are described in

calculus	the mathematics of change 变化 and of accumulation 累积
the two questions	how fast something changes at an instant , and how much accumulates over an interval
a limit	the value $f(x)$ approaches as x approaches a – and it ignores the point itself
on a graph	a hole means the limit exists and the value does not; a filled dot elsewhere means they differ
average rate of change 平均变化率	the slope of a secant; the derivative is its limit
not tested	the epsilon–delta definition is not on the AP exam

“Ignore the point itself” is the whole idea. A function can be undefined exactly where its limit is clearest.

2

Reading a limit

Defining a limit

$\lim_{x \rightarrow c} f(x) = R$ means $f(x)$ can be made
arbitrarily 任意地 close to R by taking x
sufficiently 足够 close to c – but not equal
to c .

A limit describes the **behaviour near** c ,
not the value at c .

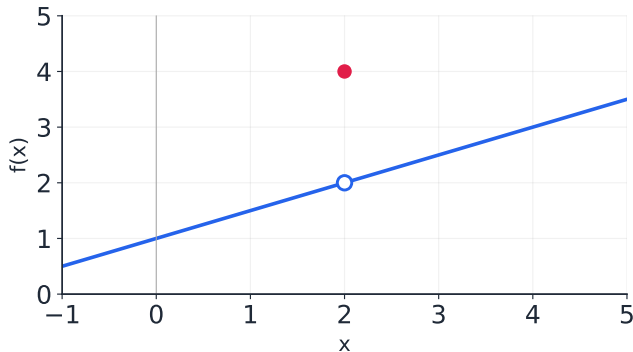
graphically 用图象

numerically 用数值

analytically 用解析式

Moving between the three is a core skill.

Read the curve and the filled point separately



Your turn, at $x = 2$

Find the left-hand limit, right-hand limit, two-sided limit and $f(2)$. Is the function continuous? Explain.

Trace nearby inputs first. Read the filled point only for the function value.

The limit is two, while the value is four

$$\lim_{x \rightarrow 2^-} f(x) = 2, \quad \lim_{x \rightarrow 2^+} f(x) = 2.$$

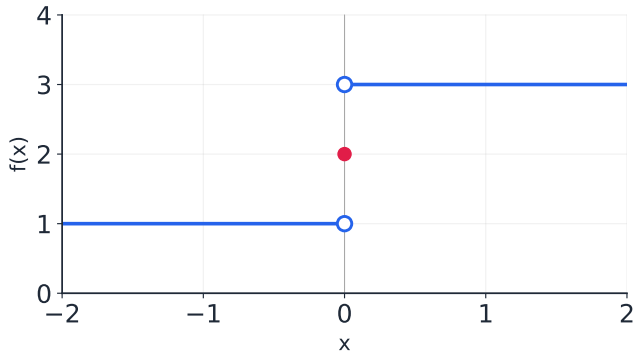
$$\lim_{x \rightarrow 2} f(x) = 2, \quad f(2) = 4.$$

Why the distinction matters

Both branches approach height 2, so the two-sided limit exists. The filled point gives 4. The function is discontinuous because the limit and value differ.

Redefining $f(2) = 2$ repairs this removable discontinuity; nearby values stay unchanged.

A filled point does not join unequal limits



Your turn, at $x = 0$

Find both one-sided limits and $f(0)$. Does a two-sided limit exist? Can changing $f(0)$ make the function continuous?

The existence of a filled point does not establish a two-sided limit.

Changing one value cannot remove a jump

$$\lim_{x \rightarrow 0^-} f(x) = 1, \quad \lim_{x \rightarrow 0^+} f(x) = 3, \quad f(0) = 2.$$

Reason from the approaching heights

Since $1 \neq 3$, the two-sided limit does not exist. This is a jump discontinuity. Any new value at 0 leaves the two nearby branches unchanged, so no single point value repairs it.

Use **left-hand limit** 左极限 and **right-hand limit** 右极限 as evidence. An **open circle** 空心圆 marks an excluded point; a **closed circle** 实心圆 gives the actual value.

When a limit does not exist

Jump 跳跃

the two sides **disagree**

$$\lim_{x \rightarrow 0} \frac{|x|}{x} \text{ DNE}$$

Unbounded 无界

grows without limit

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

Oscillates 振荡

forever near c

$$\lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ DNE}$$

Watch the **scale** 比例: a zoomed-out graph can hide behaviour near a point – confirm with algebra.

Estimating a limit from a table

Estimate $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ (it is $\frac{0}{0}$ at $x = 2$)

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4.

A table only **suggests** a value – it is a numerical **estimate**, not a proof.

3

Computing limits

The limit theorems

If both limits exist, the limit of a combination is that combination of the limits:

$$\lim(f \pm g) = \lim f \pm \lim g$$

$$\lim(fg) = \lim f \cdot \lim g$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad (\text{bottom} \neq 0)$$

$$\lim g(f(x)) = g(\lim f(x)) \quad (g \text{ continuous})$$

The practical rule: try **direct substitution** 直接代入 **first**. A real number out means you are done.

The vocabulary a limit question uses

Piecewise-defined function

A **piecewise-defined function** uses different rules on different intervals – so the one-sided limits at a join must be checked separately.

Composite

For a **composite** $f(g(x))$, the limit passes inside when f is continuous there: $\lim f(g(x)) = f(\lim g(x))$.

Radical

A **radical** expression with an indeterminate form usually needs the conjugate – multiply top and bottom by it and the difference of squares clears the root.

Continuous on an interval

A function is **continuous on an interval** when it is continuous at every point of it – and one-sided at the endpoints.

Existence theorem

An **existence theorem** – the Intermediate Value Theorem, the Extreme Value Theorem – guarantees that something exists without saying where.

The **instantaneous rate of change** is the limit of an average rate as the interval shrinks.

When substitution gives $\frac{0}{0}$

$\frac{0}{0}$ is an **indeterminate form** 未定式 – it does **not** mean the limit fails.

Rewrite into an **equivalent form** 等价形式, then substitute.

The cancelled factor is **why the graph had a hole**: the two functions agree everywhere except at c .

Three standard moves

Factor and cancel 因式分解并约分:

$$\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} (x+2) = 4$$

Multiply by the **conjugate** 共轭:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \frac{1}{2}$$

Or use a **trig identity**.

Choosing a procedure

1. substitute – a real answer means **done**

2. $\frac{0}{0}$? **rewrite**, then substitute

3. $\frac{\infty}{0}$? infinite or DNE
– **check each side**

4. $x \rightarrow \pm\infty$? compare the **dominant** 主导 terms

5. trapped between two functions? **squeeze theorem**

Choosing a method for a limit

1. substitute	if it gives a number, that is the limit – for a continuous 连续 function
2. factor and cancel	for $0/0$ caused by a common factor
3. rationalise	multiply by the conjugate when a surd causes the $0/0$
4. use identities	alternate forms of trigonometric functions often simplify what algebra cannot
5. squeeze	trap the function between two with the same limit
6. compare degrees	at infinity, the leading terms decide the end behavior 末端行为

This is a **skill** topic, not new content. And the same limit lives in a graph, a table and an algebraic form – move between all three.

The squeeze theorem

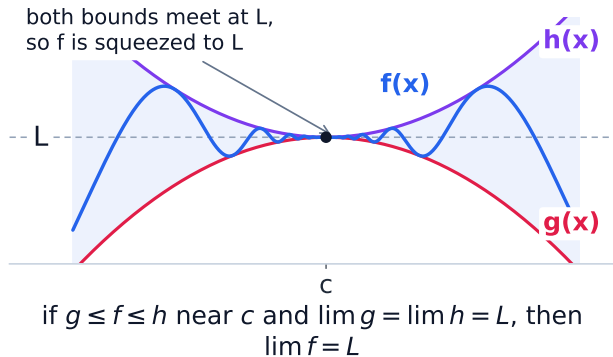
If $g(x) \leq f(x) \leq h(x)$ near c and both bounds meet at L , then f has nowhere to go:

$$\lim_{x \rightarrow c} f(x) = L$$

The two famous results proved

this way: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$



4

Continuity

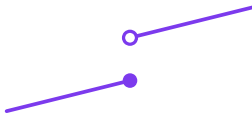
Three types of discontinuity

A function is **discontinuous** 间断 at c when its graph **breaks** there:

Removable
(hole)



Jump



Infinite
(asymptote)



f continuous at a iff $\lim_{x \rightarrow a} f(x) = f(a)$ (limit exists and equals the value)

Removable 可去间断 (a hole) ▪ **jump** 跳跃间断 (sides disagree) ▪ **infinite** 无穷间断 (an asymptote).

Classify the break, then decide whether it is repairable

Your turn, at $x = 2$

For each case, name the discontinuity and decide whether defining one point value can remove it. Give evidence about the nearby outputs.

$$A(x) = \frac{x^2 - 4}{x - 2}, \quad B(x) = \begin{cases} 1 & x < 2, \\ 3 & x \geq 2, \end{cases} \quad C(x) = \frac{1}{x - 2}.$$

For A, simplify for $x \neq 2$. For B and C, check each side separately.

A hole can be filled; a jump and an asymptote cannot

A: removable discontinuity

For $x \neq 2$, $A(x) = (x-2)(x+2)/(x-2) = x+2$. Hence $\lim_{x \rightarrow 2} A(x) = 4$. Defining $A(2) = 4$ removes the hole.

B: jump discontinuity

The one-sided limits are 1 and 3. They differ, so changing $B(2)$ cannot make them agree.

C: infinite discontinuity

As $x \rightarrow 2^-$, $C(x) \rightarrow -\infty$; as $x \rightarrow 2^+$, $C(x) \rightarrow +\infty$. There is no finite limiting value to supply at 2.

Continuity at a point – the three-part test

f is **continuous** 连续 at $x = c$ exactly when **all three** hold:

$f(c)$ **exists**

$\lim_{x \rightarrow c} f(x)$ **exists**

they **agree**: $\lim_{x \rightarrow c} f(x) = f(c)$

The point is there, the limit is there, and the two **agree**. If any one fails, f is discontinuous.

Learn it as a **checklist** – it is the backbone of nearly every continuity question.

Continuity over an interval, and repairing a hole

You rarely check point by point – whole families are continuous **on their domains**:

Polynomial, rational, power, **exponential** 指数, **logarithmic** 对数 and **trigonometric** 三角 functions are continuous at every point of their domains.

So a rational function is continuous everywhere **except** where its denominator is zero.

Removing a discontinuity

If the limit exists at a hole, **redefine** the function there to equal the limit – the graph is repaired.

For a **piecewise** 分段函数 function at a boundary, solve for the **parameter** 参数 that makes the pieces **meet**:

$$\lim_{x \rightarrow c^-} f = \lim_{x \rightarrow c^+} f = f(c).$$

5

Asymptotes

Infinite limits and vertical asymptotes

Where a **non-zero number** is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** 垂直渐近线.

Always check **each side**: the two sides can shoot **opposite** ways.

End behaviour, from the leading terms

Relative magnitudes

At infinity only the **relative magnitudes** matter: exponentials beat any polynomial, and a polynomial beats any logarithm.

Equal degrees

When numerator and denominator have the same degree, the limit is the **ratio of the leading coefficients** – and that ratio is the horizontal asymptote.

Unequal degrees

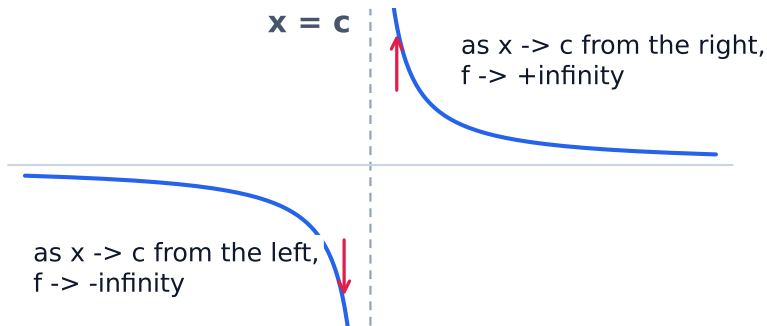
Numerator smaller: the limit is 0. Numerator larger: it grows without bound, and polynomial division gives the slant asymptote.

Naming it

That long-run **behavior** – the **end behavior** – is what a limit at infinity reports.

Divide every term by the highest power in the denominator. Every rational end-behaviour limit falls out of that one step.

Infinite limits and vertical asymptotes, in detail



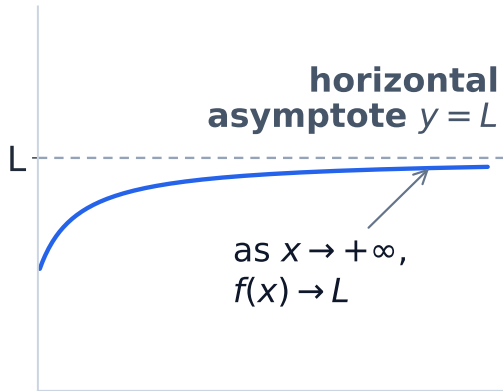
vertical asymptote $x = c$ when $\lim_{x \rightarrow c^+} f(x) = \pm \infty$ or
 $\lim_{x \rightarrow c^-} f(x) = \pm \infty$

Where a **non-zero number** is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** 垂直渐近线.

Limits at infinity and horizontal asymptotes

Let the **input** grow: **end behaviour** 末端行为 as $x \rightarrow \pm\infty$. If outputs settle toward L , then $y = L$ is a **horizontal asymptote** 水平渐近线.

Far out, an **exponential** beats any **polynomial**, and a polynomial beats any logarithm.



Rational functions – compare the degrees

top < bottom
limit is 0
(asymptote $y = 0$)

top = bottom
the **ratio of
leading coeffi-
cients** 首项系数之比

top > bottom
unbounded –
no horizon-
tal asymptote

“As $t \rightarrow \infty$, where does the rate settle?” is answered with a **limit at infinity**.

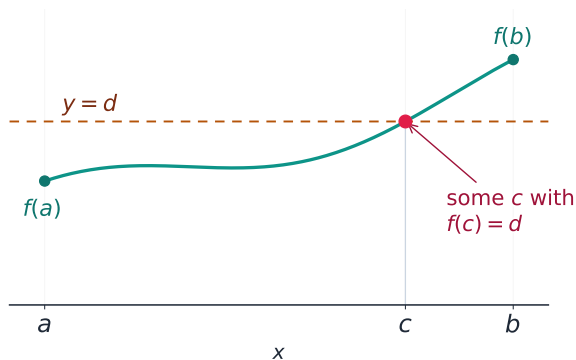
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IVT

The Intermediate Value Theorem

If f is **continuous** on $[a, b]$ and d lies **between** $f(a)$ and $f(b)$, then $f(c) = d$ for at least one c in (a, b) .

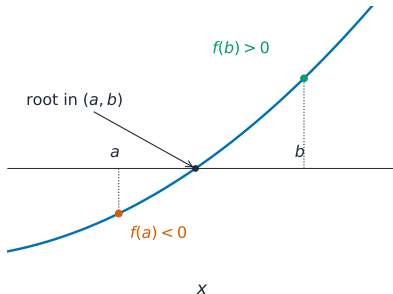
An unbroken curve **cannot skip** a height between its endpoints.



Exam skill – justifying with the IVT

These appear almost every year. A full-credit justification has **three moves**:

- 1 state continuity on $[a, b]$
- 2 show d is trapped: $f(a) < d < f(b)$
- 3 conclude by name: “by the IVT there is a c in (a, b) with $f(c) = d$ ”



Skipping the continuity statement – or not showing d is between the endpoints – **loses the point**.

Worked example – a hole at $x = 3$

Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

Direct substitution gives $\frac{0}{0}$, so factor:

$$\frac{(x - 3)(x + 3)}{x - 3} = x + 3 \quad (x \neq 3)$$

$$\lim_{x \rightarrow 3} (x + 3) = 6.$$

The graph has a **removable discontinuity** at $x = 3$ – the limit exists even though the function is **undefined** there.

Defining Limits and Using Limit Notation



A smooth bridge curve: limits describe the value a graph approaches as we zoom in

Check yourself, in detail

Q1 – the $\frac{0}{0}$ form

Substitution gives $\frac{0}{0}$, which is a signal to **rewrite**, not a failure.
Factor and cancel: $\frac{(x-3)(x+3)}{x-3} = x+3$ for $x \neq 3$, so the limit is $3+3=6$.

Q2 – limit versus value

No. The two agree **only** when f is continuous at c . The hole graph is the counter-example: the curve approaches one height while the filled dot sits at another.

Q3 – the three parts

$f(c)$ exists; $\lim_{x \rightarrow c} f(x)$ exists; the two are **equal**. Each way of failing names a discontinuity – a missing or misplaced value, one-sided limits that disagree, or an unbounded side.

Q4 – the IVT

Continuity on the **closed** interval $[a, b]$, and the target d trapped **between** $f(a)$ and $f(b)$. Then conclude by name – an existence claim with no theorem behind it earns nothing.

All four turn on one distinction: a **limit** is about the approach, a **value** is about the point. Continuity is the sentence that says the two coincide.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ – the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes where the denominator (not the numerator) is zero.

7

Recap

Unit 1 – key takeaways

1

A limit

what f approaches near c – not the value at c

2

Computing

substitute first; on $0/0$ factor, cancel or rationalise

3

Continuity

value exists, limit exists, and the two agree

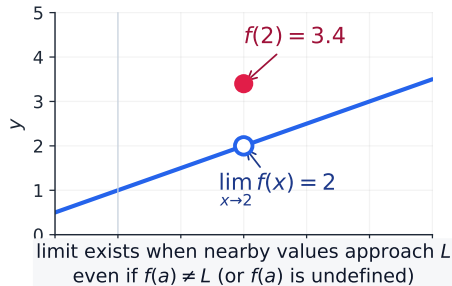
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The IVT

continuous + d trapped $\Rightarrow$ some c hits d

Check yourself

- 1 Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.
- 2 Must $\lim_{x \rightarrow c} f(x)$ equal $f(c)$?
- 3 Name the three parts of the continuity test.
- 4 Which two conditions must an IVT justification state?



Answers 1. 6 2. no – only if f is continuous at c 3. $f(c)$ exists, the limit exists, they agree 4. continuity, and d between $f(a)$ and $f(b)$