Course Companion

AP Calculus AB

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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Limits and Continuity

AP Calculus AB

Introducing Calculus: Can Change Occur at an Instant?

Calculus is the mathematics of **change** 变化 and of **accumulation** 累积. It answers two big questions: how fast is something changing *right now*, and how much has piled up *so far*? Unit 1 builds the one tool both questions rest on – the **limit** 极限.

Start with a puzzle. A car's speedometer reads 60 km/h. What does that mean *at a single instant* 瞬间? Speed is distance over time. But at one instant no time passes and no distance is covered, so the fraction looks like $\frac{0}{0}$ – undefined.

- The **average rate of change** 平均变化率 uses a whole *interval* 区间: the change in one quantity divided by the change in another. It divides by zero, and so is undefined, when the change in the input would be zero.
- The **instantaneous rate of change** 瞬时变化率 is what we want *at a point*. It is the value the average rate **approaches** 趋近 as the interval shrinks toward zero length.

The clever move is not to plug in zero (undefined), but to watch what the average rate *approaches* as the interval gets smaller and smaller. That approaching value is a **limit**. So calculus lets us describe change at an instant – as a limit of average rates over ever-shorter intervals. This one idea powers the **derivative** 导数 (Unit 2) and, run in reverse, the **integral** 积分 (Unit 6). Everything else in this unit defines limits carefully and computes them reliably.

Defining Limits and Using Limit Notation

A smooth bridge curve: limits describe the value a graph approaches as we zoom in

Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c – **but not equal to** c. We write

$$\lim_{x\to c} f(x) = R$$

and read it: "the limit of $f(x)$, as x approaches c, equals R."

The last words are the heart of a limit: it describes the **behavior** 行为 of f *near* c, not the value *at* c. The function may be undefined at c, or defined but equal to something else – the limit does not care.

A limit can be shown in three ways: **graphically** 用图象, **numerically** 用数值 (a table), and **analytically** 用解析式 (algebra). Learning to move between these representations is a core skill.

*(Note: the epsilon-delta definition of a limit is **not** tested on the AP Exam, so this handout does not use it.)*

Estimating Limit Values from Graphs

A graph is often the fastest way to read a limit. To find $\lim_{x\to c} f(x)$, run your finger along the curve toward $x = c$ from each side and ask: what height is the curve heading for?

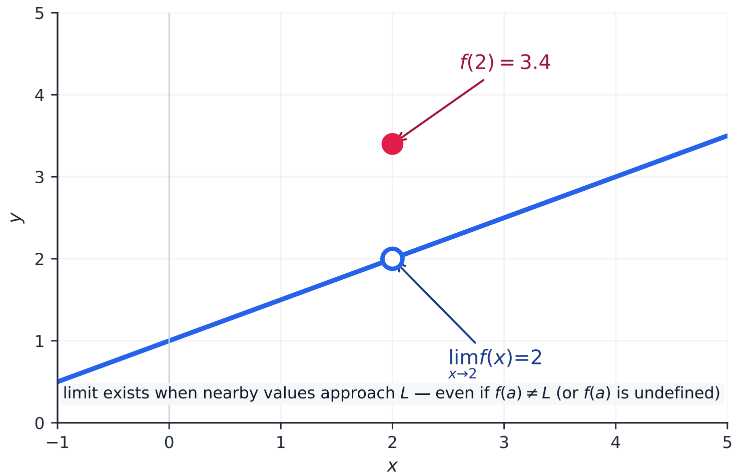
- Trace from the **left** (inputs smaller than c): this gives the **left-hand limit** 左极限, $\lim_{x\to c^-} f(x)$.

- Trace from the **right** (inputs larger than c): this gives the **right-hand limit** 右极限, $\lim_{x\to c^+} f(x)$.
- These are the **one-sided limits** 单侧极限. If both head to the *same* height R, then the two-sided limit exists and $\lim_{x\to c} f(x) = R$.

Crucially, **ignore the point itself**. Graphs mark the difference between the limit and the value:

- An **open circle** 空心圆 marks a height the curve *approaches* but does not reach – a "hole" 空洞.
- A **closed circle** 实心圆 marks the actual value $f(c)$.

So a curve may approach $R = 3$ from both sides (limit is 3) while a filled dot sits at height 5 (value $f(c) = 5$). The limit is 3; the two need not match.



The open circle is the height the curve approaches (the limit); the filled dot is the actual value $f(2)$ – they need not agree.

A limit **does not exist** (often written **DNE**) when the two sides disagree (a **jump** 跳跃), when the function is **unbounded** 无界 (grows without limit), or when it **oscillates** 振荡 forever near c. For example:

$$\lim_{x\to 0}\frac{1}{x^2} = \infty, \qquad \lim_{x\to 0}\frac{|x|}{x}\ \text{DNE}, \qquad \lim_{x\to 0}\sin\left(\frac{1}{x}\right)\ \text{DNE}.$$

Watch the **scale** 比例 of a graph: a zoomed-out picture can hide important behavior near a point, so confirm with algebra when you can.

Estimating Limit Values from Tables

When you have data or a formula but no picture, a **table** 表格 of values estimates a limit numerically. Choose inputs that creep toward c from both sides and watch the outputs.

For example, to estimate $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ (which is $\frac{0}{0}$ at $x = 2$):

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4, so we estimate the limit is 4. A table only *suggests* a value – it is a numerical estimate, not a proof.

Determining Limits Using Algebraic Properties of Limits

Most limits are found **analytically** using **limit theorems** 极限定理. If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist, the limit of a combination is the same combination of the limits:

- **Sum / difference:** $\lim_{x \to c} \big[f(x) \pm g(x)\big] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
- **Product:** $\lim_{x \to c} \big[f(x)\, g(x)\big] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
- **Quotient:** $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided the bottom limit is not 0.
- **Composite** 复合函数: if g is continuous at $\lim_{x \to c} f(x)$, then $\lim_{x \to c} g\big(f(x)\big) = g\Big(\lim_{x \to c} f(x)\Big)$.

The practical rule: for a function built from polynomials, roots, and the like, first try **direct substitution** 直接代入 – put $x = c$ in. If you get a real number, that is the limit. One-sided limits obey the same theorems, read from one direction only.

Determining Limits Using Algebraic Manipulation

Direct substitution sometimes gives the **indeterminate form** 未定式 $\frac{0}{0}$. This does **not** mean the limit fails – it means you must rewrite the function into an **equivalent form** 等价形式 that removes the trouble, then substitute. Three standard moves:

- **Factor and cancel** 因式分解并约分 a **rational function** 有理函数. Example: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$.
- Multiply by the **conjugate** 共轭 to simplify a **radical** 根式. Example: $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \to 0} \frac{x}{x\big(\sqrt{x+1} + 1\big)} = \frac{1}{2}$.
- **Use alternate forms** of trigonometric functions (identities) to simplify.

The cancelled factor is why the original graph had a hole: the two functions agree everywhere except at $x = c$, so they share the same limit there.

Selecting Procedures for Determining Limits

This is a skill topic, not new content: choose the right tool for the limit in front of you.

1. **Try direct substitution first.** A real answer means you are done.
2. Getting $\frac{0}{0}$? **Rewrite** – factor and cancel, or use the conjugate, or a trig identity – then substitute.
3. A **non-zero number over** 0 (like $\frac{5}{0}$)? The limit is infinite or DNE – check the sign from each side (see vertical asymptotes below).
4. As $x \to \pm\infty$? Compare the **dominant** 主导 terms (see limits at infinity).
5. Trapped between two functions? The **squeeze theorem** may apply.

Determining Limits Using the Squeeze Theorem

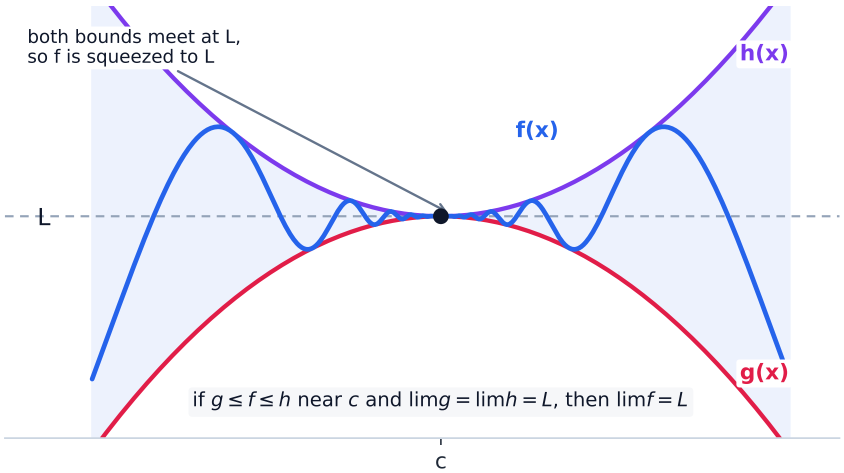
The **squeeze theorem** 夹逼定理 (also called the sandwich theorem) finds a limit by trapping the function between two others. If $g(x) \le f(x) \le h(x)$ near c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then f is squeezed to the same place: $\lim_{x \to c} f(x) = L$.

The two famous results proved this way, both used throughout calculus, are:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$



The wiggly f is trapped between g and h. Because both bounds meet at L, f has nowhere to go but L.

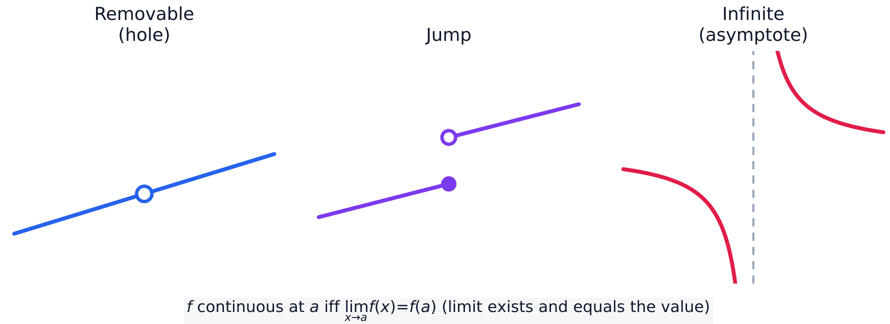
Connecting Multiple Representations of Limits

Another skill topic: the same limit lives in a graph, a table, and an algebraic form, and you should be able to translate between them. A graph shows the shape and any holes or jumps; a table gives numerical evidence; algebra gives an exact value and a reason. Strong answers use one representation to *confirm* another.

Exploring Types of Discontinuities

A function is **discontinuous** 间断 at c when its graph "breaks" there. There are three types:

- **Removable discontinuity** 可去间断 – a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** 跳跃间断 – the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** 无穷间断 – the function blows up to $\pm\infty$ at a **vertical asymptote** 垂直渐近线.



Three ways a graph can break: a removable hole, a jump, and an infinite (asymptote) discontinuity.

Defining Continuity at a Point

Continuity is defined by a three-part test. A function f is **continuous** 连续 at $x = c$ exactly when all three hold:

$$\boxed{f(c) \text{ exists} \quad \text{and} \quad \lim_{x\to c} f(x) \text{ exists} \quad \text{and} \quad \lim_{x\to c} f(x) = f(c)}$$

In words: the point is there, the limit is there, and the two agree. If any one fails, f is discontinuous at c. This test is the backbone of nearly every continuity question, so learn it as a checklist.

Confirming Continuity over an Interval

A function is **continuous on an interval** 在区间上连续 if it is continuous at *every* point of that interval. You rarely check point by point, because whole families are continuous on their domains:

> **Polynomial, rational, power, exponential 指数, logarithmic 对数, and trigonometric 三角 functions are continuous at every point of their domains.**

So a rational function is continuous everywhere *except* where its denominator is zero; $\ln x$ is continuous for $x > 0$; and so on. Knowing this lets you declare continuity quickly and correctly.

Removing Discontinuities

If the **limit exists** at a hole, the discontinuity is **removable**: redefine the function at that one point to equal the limit, and the graph is repaired. Formally, set the missing value to $\lim\limits_{x \to c} f(x)$.

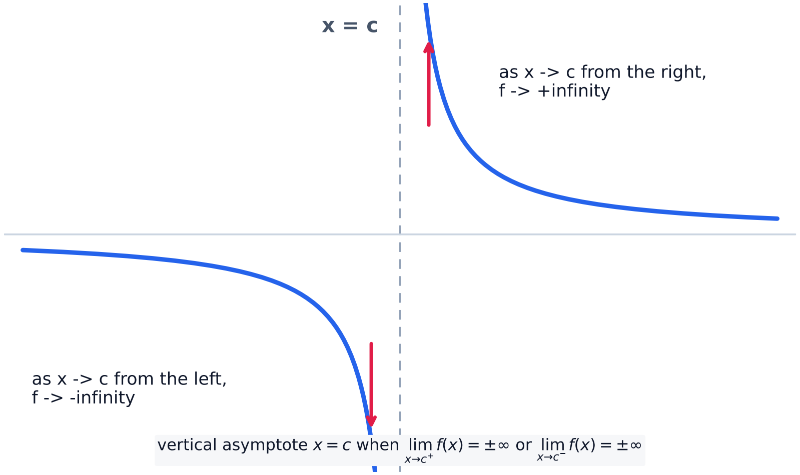
For a **piecewise-defined function** 分段函数, continuity at a boundary $x = c$ needs the two pieces to *meet*: the left piece's value, the right piece's value, and $f(c)$ must all be equal. This is a common exam setup – you solve for a **parameter** 参数 (an unknown constant) that makes the pieces match:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

Connecting Infinite Limits and Vertical Asymptotes

The idea of a limit extends to **infinite limits** 无穷极限. When a function grows without bound near $x = c$, we write $\lim_{x \to c} f(x) = \pm\infty$. This describes a **vertical asymptote** at $x = c$: the graph hugs the vertical line $x = c$ and shoots off toward $\pm\infty$.

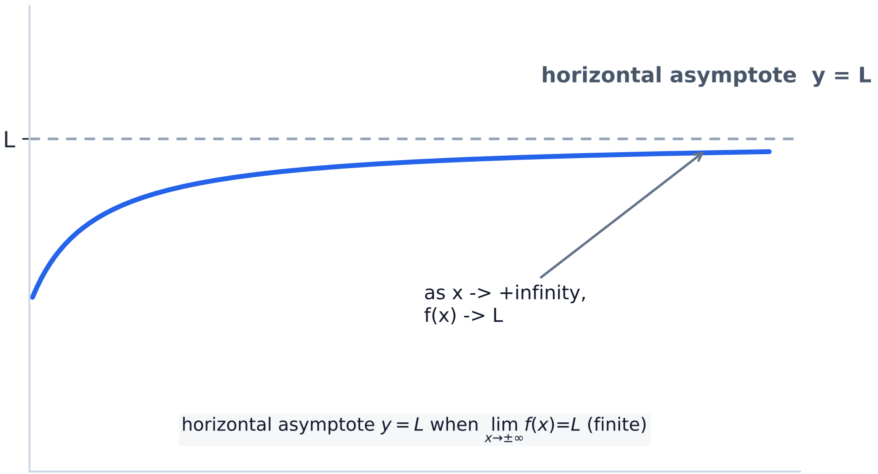
This happens where a **non-zero number is divided by something approaching** 0, such as at a zero of a denominator that does not cancel. Always check each side separately – the two sides can shoot **opposite** ways (one to $+\infty$, one to $-\infty$).



Near a vertical asymptote the graph hugs the line $x = c$. Here the left side falls to $-\infty$ and the right side climbs to $+\infty$, so the two-sided limit does not exist.

Connecting Limits at Infinity and Horizontal Asymptotes

We can also let the **input** grow: **limits at infinity** 无穷远处的极限 describe the **end behavior** 末端行为 of a function as $x \to \pm\infty$. If the outputs settle toward a finite value L, then $y = L$ is a **horizontal asymptote** 水平渐近线.



As $x \to +\infty$ the curve flattens toward the line $y = L$. The horizontal asymptote records this end behavior.

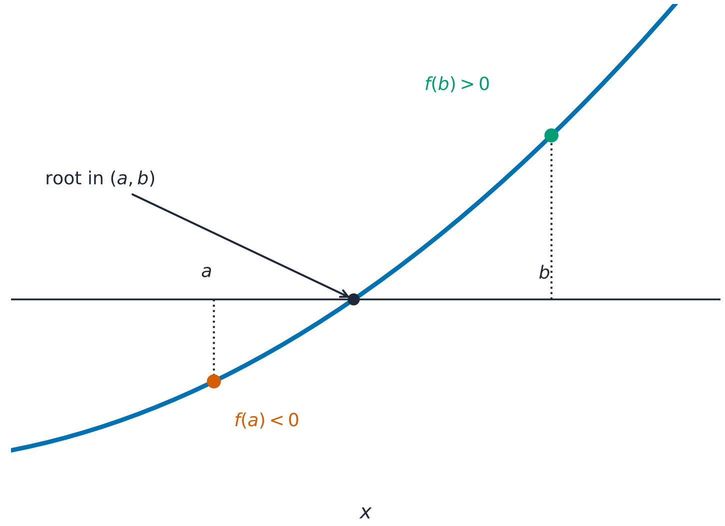
For a rational function, compare the **degrees** 次数 of the top and bottom:

- top degree $<$ bottom degree $\Rightarrow$ limit is 0 (asymptote $y = 0$);
- top degree $=$ bottom degree $\Rightarrow$ limit is the **ratio of the leading coefficients** 首项系数之比;
- top degree $>$ bottom degree $\Rightarrow$ the function is unbounded (no horizontal asymptote).

More generally, we compare the **relative magnitudes** 相对大小 (relative growth rates) of functions: far out, an exponential beats any polynomial, and a polynomial beats any logarithm. On the exam, "as $t \to \infty$, which quantity is larger/where does the rate settle?" is answered with a limit at infinity.

Working with the Intermediate Value Theorem (IVT)

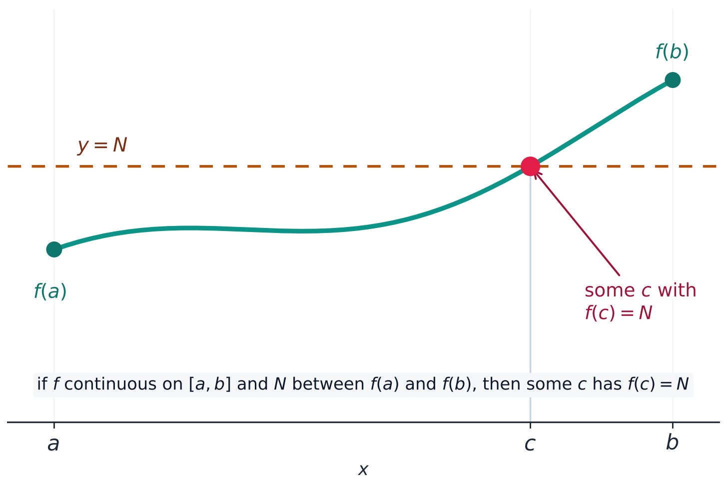
The **Intermediate Value Theorem** 介值定理 is an **existence theorem** 存在性定理 – it guarantees a value exists without telling you where:



Opposite signs of $f(a)$ and $f(b)$ trap a root between a and b

> If f is **continuous** on the closed interval $[a, b]$, and d is any number **between** $f(a)$ and $f(b)$, then there is **at least one** number c in (a, b) with $f(c) = d$.

An unbroken curve cannot skip a height between its endpoints – it must pass through every one.



A continuous curve from $(a, f(a))$ to $(b, f(b))$ must cross every height d in between at least once.

Exam skill – how to justify with the IVT. These questions appear almost every year (for example, "Must there be a value c with $R(c) = 155$?" or "Is there a time when $r'(t) = -6$?"). A full-credit justification has three moves:

1. **State continuity.** Say the function is continuous on $[a, b]$ (often because it is differentiable, or given continuous).
2. **Show d is trapped.** Compute the two endpoint values and show the target d lies **between** them, e.g. $f(a) < d < f(b)$.
3. **Conclude by name.** "By the Intermediate Value Theorem, there is a c in (a, b) with $f(c) = d$."

Skipping the continuity statement, or not showing d is between the endpoints, loses the point – the theorem *requires* both conditions.

Worked example. Evaluate $\lim_{x\to 3} \dfrac{x^2 - 9}{x - 3}$. Direct substitution gives $\frac{0}{0}$ (indeterminate), so factor: $\dfrac{(x-3)(x+3)}{x-3} = x+3$ for $x \neq 3$. Then $\lim_{x\to 3}(x+3) = 6$. The graph has a **removable discontinuity** (a hole) at $x = 3$ — the limit exists even though the function is undefined there.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ — the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes where the denominator (not the numerator) is zero.

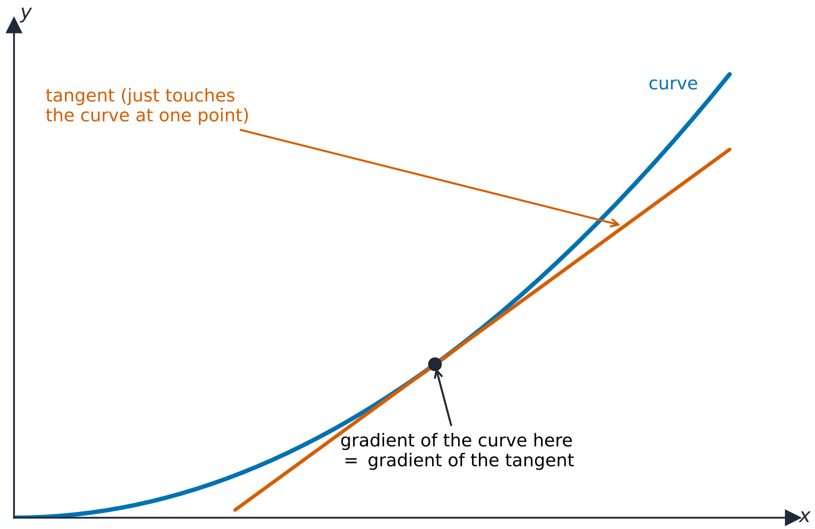
Differentiation: Definition and Fundamental Properties

AP Calculus AB

Average and Instantaneous Rates of Change

A roller coaster on a lift hill: the derivative measures instantaneous rate of change of position

Unit 1 built the limit. Unit 2 uses it to define the **derivative** 导数 – the exact rate of change at a point.



The instantaneous rate of change is the gradient of the tangent at a point

Over an interval, the average rate of change is a **difference quotient** 差商. Two equivalent forms appear:

$$\frac{f(a+h)-f(a)}{h} \quad \text{and} \quad \frac{f(x)-f(a)}{x-a}.$$

The first uses a step of size h from a; the second uses two points x and a. Both compute $\frac{\text{change in output}}{\text{change in input}}$ over the interval.

The **instantaneous** 瞬时 rate of change at $x = a$ is what the difference quotient approaches as the interval shrinks to zero. This limit *is* the derivative at a, written $f'(a)$:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a},$$

provided the limit exists.

Defining the Derivative and Its Notation

Let the point a vary and the derivative becomes a **new function**:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

This is the **definition of the derivative** (sometimes called differentiating "by first principles" 用定义求导). Its value at each x is the instantaneous rate of change there.

Common **notations** 记号 for the derivative of $y = f(x)$ are:

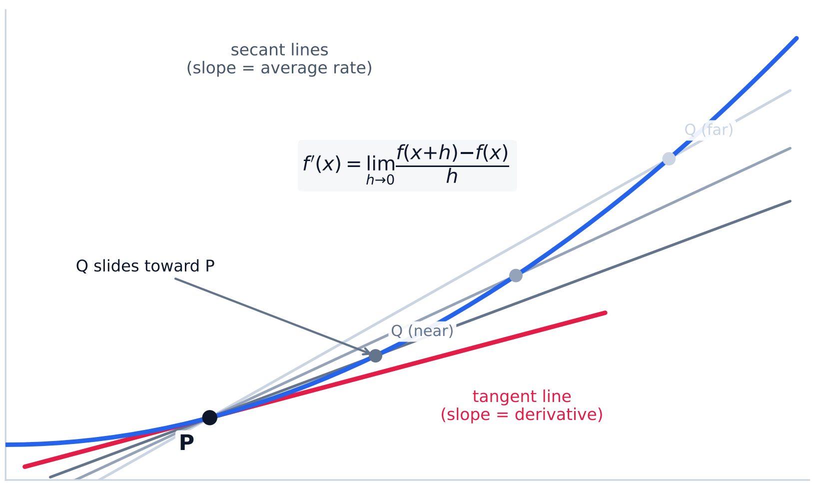
$$\frac{dy}{dx}, \qquad f'(x), \qquad y'.$$

The derivative can be represented graphically, numerically, analytically, and verbally – be ready to move between them.

Geometric meaning. The derivative at a point is the **slope** 斜率 of the **tangent line** 切线 to the graph there. So the tangent line at $x = a$ passes through $(a, f(a))$ with slope $f'(a)$:

$$y - f(a) = f'(a)\,(x - a).$$

Writing this line is a routine exam task, so keep the point-slope form ready.



As Q slides toward P, each secant's slope (an average rate) approaches the tangent's slope – the derivative $f'(a)$.

Estimating a Derivative at a Point

You do not always have a formula. When a function is given by a **table** 表格 or a graph, estimate the derivative $f'(a)$ with a difference quotient over a **small interval around** a. A table with values on both sides of a gives the best estimate:

$$f'(a) \approx \frac{f(b) - f(c)}{b - c}, \qquad \text{where } c < a < b \text{ are the closest table inputs.}$$

Technology (a calculator) can also estimate a derivative at a point.

Exam skill (appears almost every year). Questions such as "Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$" ask for exactly this difference quotient. Show the setup:

$$M'(7.5) \approx \frac{M(10) - M(5)}{10 - 5}.$$

Full credit needs the numbers plugged in **and** the correct **units** 单位 (output units per input unit), since these come from real-world models.

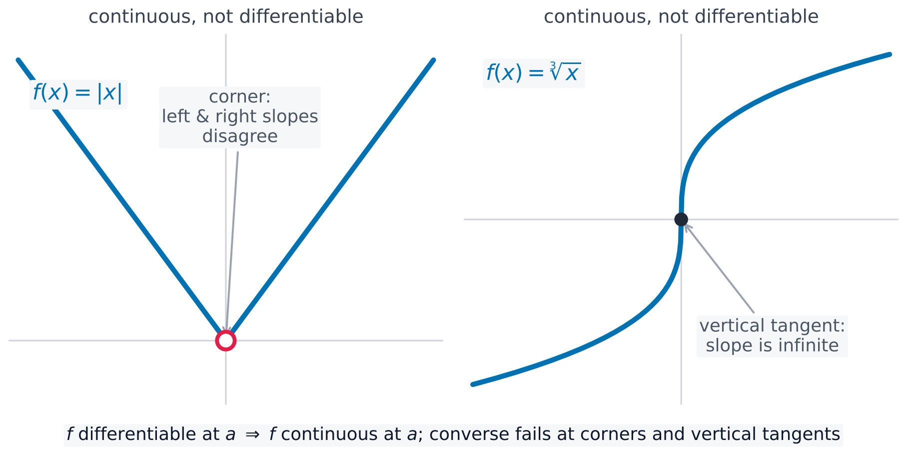
When Does a Derivative Exist?

Differentiability is stronger than continuity. The key relationship:

> **If f is differentiable 可导 at a point, then f is continuous 连续 there.**

So differentiability *implies* continuity. The reverse is false: a continuous function can fail to be differentiable. Two ways this happens:

- A **corner** 尖点: the left and right difference-quotient limits disagree, as with $f(x) = |x|$ at $x = 0$.
- A **vertical tangent** 垂直切线: the slope is infinite (no real number), as with $f(x) = \sqrt[3]{x}$ at $x = 0$.



Two ways a continuous function is not differentiable: a corner and a vertical tangent

Also, a point outside the domain of f cannot be in the domain of f'. Use the contrapositive on the exam: **if f is not continuous at a, then f is not differentiable at a.**

The Power Rule

From here we use **rules** instead of the limit definition each time. The **power rule 幂法则** handles any power of x:

$$\frac{d}{dx}\, x^r = r\, x^{r-1} \qquad \text{for any real } r.$$

It works for whole-number powers, negative powers ($\frac{1}{x} = x^{-1}$), and roots ($\sqrt{x} = x^{1/2}$) – rewrite as a power first, then apply the rule.

Constant, Sum, Difference, and Constant Multiple Rules

These rules let you differentiate term by term:

- **Constant:** $\dfrac{d}{dx}\, k = 0$ (a constant does not change).
- **Constant multiple 常数倍:** $\dfrac{d}{dx}\big[k\, f(x)\big] = k\, f'(x)$.
- **Sum / difference:** $\dfrac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x)$.

Combined with the power rule, they differentiate any **polynomial 多项式** term by term. Example:

$$\frac{d}{dx}\big(4x^3 - 5x + 7\big) = 12x^2 - 5.$$

Derivatives of sin x, cos x, e^x, and ln x

Learn these four building-block derivatives by heart:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0).$$

Note the minus sign on the derivative of cosine, and that e^x is its own derivative.

A limit that is really a derivative (LIM-3.A.1). Sometimes a limit is secretly the definition of a known derivative. If you recognize

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

for a function f whose derivative you know, just evaluate $f'(a)$. For example, $\displaystyle\lim_{h\to 0} \frac{\sin\!\left(\frac{\pi}{2} + h\right) - 1}{h} = \frac{d}{dx}\sin x\Big|_{x=\pi/2} = \cos\frac{\pi}{2} = 0.$

The Product Rule

A product of two functions is **not** differentiated by multiplying the derivatives. Use the **product rule 乘积法则**:

$$\frac{d}{dx}\big[u\, v\big] = u'v + uv'.$$

"Derivative of the first times the second, plus the first times the derivative of the second." Example:

$$\frac{d}{dx}\big(x^2 e^x\big) = 2x\, e^x + x^2 e^x.$$

Exam questions often build a new function from given pieces, e.g. $k'(x) = \big(f(x)\big)^2 g(x)$, and ask you to combine rules while reading values from a table.

The Quotient Rule

For a quotient, use the **quotient rule 商法则**:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}.$$

"Bottom times derivative of top, minus top times derivative of bottom, all over bottom squared." The order matters because of the minus sign, so write the numerator carefully. Example:

$$\frac{d}{dx}\left(\frac{x}{\cos x}\right) = \frac{1\cdot\cos x - x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos x + x\sin x}{\cos^2 x}.$$

Derivatives of tan x, cot x, sec x, and csc x

The remaining trigonometric derivatives are not memorized separately – you rewrite them with **identities** 恒等式 and apply the quotient (or product) rule. For instance, $\tan x = \dfrac{\sin x}{\cos x}$, so the quotient rule gives

$$\frac{d}{dx}\tan x = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The same method (writing $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$) gives $-\csc^2 x$, $\sec x \tan x$, and $-\csc x \cot x$.

Higher-order derivatives. Differentiating f' again gives the **second derivative** 二阶导数 $f''(x)$ (or $\frac{d^2y}{dx^2}$) – the rate of change of the rate of change. An exam part like "Find $k''(3)$" just means differentiate twice, then substitute. You can also estimate a second derivative from a table by applying the average-rate-of-change method to the f' values.

Worked example. Differentiate $f(x) = x^2 \sin x$ with the **product rule** $(uv)' = u'v + uv'$: take $u = x^2$ ($u' = 2x$) and $v = \sin x$ ($v' = \cos x$), giving $f'(x) = 2x \sin x + x^2 \cos x$. Always read the *structure* first — a product needs the product rule, not the power rule applied to each factor separately.

Exam tips

- The derivative is the **slope of the tangent** — the limit of the secant slope $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$.
- Memorise the rules: power, product, quotient, and the derivatives of sin, cos, e^x, and $\ln x$.
- **Differentiability implies continuity**, but not the reverse (a corner or cusp is continuous yet not differentiable).
- Distinguish average rate of change (secant slope over an interval) from instantaneous rate (the derivative at a point).
- Give a tangent-line equation as $y - f(a) = f'(a)(x - a)$.

Differentiation: Composite, Implicit, and Inverse Functions

AP Calculus AB

The Chain Rule

Unit 2 differentiated single functions. Unit 3 differentiates functions built *inside* other functions. The **chain rule** 链式法则 differentiates a **composite function** 复合函数 $f\big(g(x)\big)$:

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x).$$

"Derivative of the outer function (leaving the inside alone), times the derivative of the inside." The inner derivative $g'(x)$ is the piece students forget, so always ask "what is the inside, and what is its derivative?" Example:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2) \cdot 2x.$$

In Leibniz notation, with $y = f(u)$ and $u = g(x)$, the rule reads $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ – the intermediate du appears to "cancel." Exam questions often give a table for f, g, f', g' and ask for $h'(a)$ where $h(x) = f\big(g(x)\big)$; evaluate $f'\big(g(a)\big) \cdot g'(a)$ by reading values.

Worked example. Differentiate $h(x) = (2x^2 + 1)^5$. The outer function is "(something)^5", the inner is $2x^2 + 1$:

$$h'(x) = 5(2x^2+1)^4 \cdot \frac{d}{dx}(2x^2+1) = 5(2x^2+1)^4 \cdot 4x = 20x(2x^2+1)^4.$$

Matryoshka dolls: the chain rule differentiates nested functions layer by layer — outer first, then the next inside

Implicit Differentiation

Some curves are defined **implicitly** – by an equation in x and y that is not solved for y, such as $x^2 + y^2 = 25$. **Implicit differentiation** 隐函数求导 finds $\frac{dy}{dx}$ without solving for y first. It is just the chain rule, treating y as a function of x.

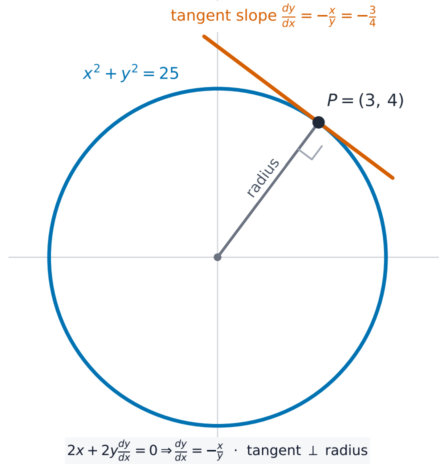
The method: differentiate **both sides** with respect to x; every time you differentiate a y-term, multiply by $\frac{dy}{dx}$ (the chain rule); then solve algebraically for $\frac{dy}{dx}$. For $x^2 + y^2 = 25$:

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

Worked example. Find the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. From above $\frac{dy}{dx} = -\frac{x}{y} = -\frac{3}{4}$ there, so the tangent is

$$y - 4 = -\tfrac{3}{4}(x - 3).$$

Notice the tangent is perpendicular to the radius, as geometry promises – a good sanity check.



Implicit differentiation gives the tangent to a circle, perpendicular to the radius

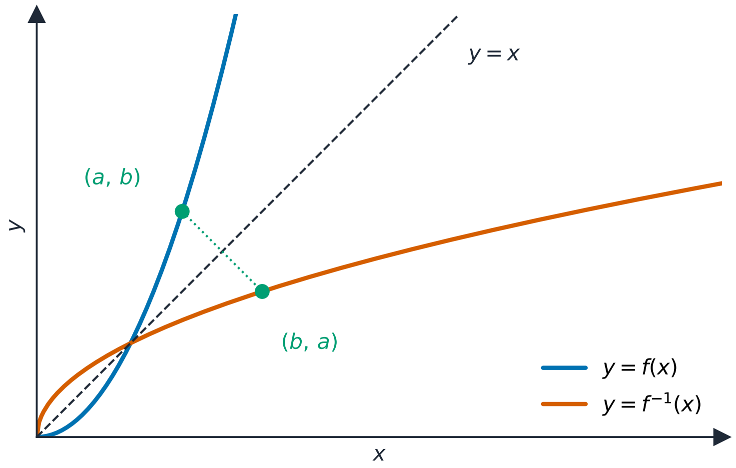
Exam skill – "Show that $\frac{dy}{dx} = \ldots$". This exact prompt appears most years (e.g. "Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$"). Because the target is given, you must show **every algebra step** cleanly: differentiate both sides, use the product/chain rules on mixed xy terms, collect all $\frac{dy}{dx}$ terms on one side, factor, and divide. A correct final line that skips the algebra earns little. Follow-up parts then ask for a tangent line, or where the tangent is horizontal ($\frac{dy}{dx} = 0$, so the numerator is 0) or vertical (the denominator is 0).

Differentiating Inverse Functions

If g is the **inverse function** 反函数 of f (so $f(g(x)) = x$), the chain rule links their derivatives:

$$g'(x) = \frac{1}{f'\big(g(x)\big)}, \qquad \text{provided } f'\big(g(x)\big) \neq 0.$$

In words: the derivative of the inverse at a point is the **reciprocal** 倒数 of the derivative of the original function at the *matching* point. A common exam setup gives a table and a point (a, b) on f (so (b, a) is on g), then asks for $g'(b) = \dfrac{1}{f'(a)}$.



The inverse function is the mirror image of the function in the line $y = x$

Worked example. Suppose $f(2) = 5$ and $f'(2) = 3$, and g is the inverse of f. Because $(2, 5)$ lies on f, the point $(5, 2)$ lies on g, and

$$g'(5) = \frac{1}{f'(2)} = \frac{1}{3}.$$

The graphs of f and g are mirror images across $y = x$, so their slopes at matching points are reciprocals.

Differentiating Inverse Trigonometric Functions

The same idea gives the derivatives of the **inverse trigonometric functions** 反三角函数. The three you should know:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1 + x^2}, \quad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Combine these with the chain rule when the input is itself a function, e.g. $\dfrac{d}{dx}\arctan(3x) = \dfrac{3}{1 + 9x^2}$.

Selecting Procedures for Calculating Derivatives

A skill topic: real derivatives mix several rules, so read the structure of the expression first, from the outside in.

- Is it a **sum**? Differentiate term by term.
- A **product** or **quotient**? Apply that rule, and expect to use the chain rule inside.
- A **composite** (something inside something)? Chain rule.
- Given **implicitly**? Implicit differentiation.

Name the outermost operation, apply its rule, and recurse inward. Neatness prevents the sign and bookkeeping errors that cost marks.

Calculating Higher-Order Derivatives

Differentiating f' produces the **second derivative** 二阶导数 f''; repeating gives higher-order derivatives. The notations:

$$f''(x) = \frac{d^2y}{dx^2} = y'', \qquad \text{and in general} \qquad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

The second derivative measures how the *slope* is changing; it drives **concavity** 凹凸性 and acceleration in later units. To find f'' implicitly, differentiate the expression for $\dfrac{dy}{dx}$ again (with the quotient and chain rules), then substitute $\dfrac{dy}{dx}$ back in.

Exam tips

- Use the **chain rule** for composite functions — differentiate the outside, then multiply by the derivative of the inside (the most-forgotten factor).
- For **implicit** differentiation, differentiate both sides with respect to x and attach $\frac{dy}{dx}$ each time y is differentiated, then solve.

- Get the **second derivative** by differentiating twice (velocity $\to$ acceleration).
- Combine rules carefully in layered expressions (chain inside product, etc.).
- The inverse function's graph is the reflection in $y = x$; its slope is the reciprocal of the original's at the matching point.

Contextual Applications of Differentiation

AP Calculus AB

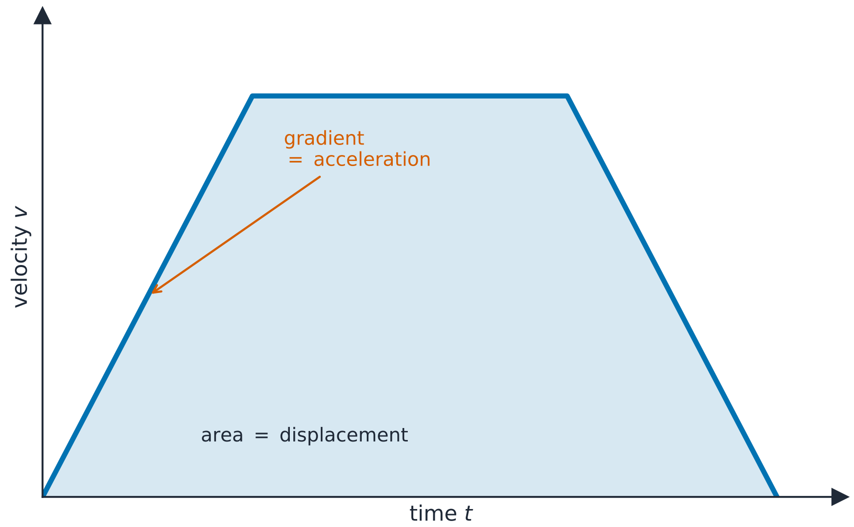
Interpreting the Derivative in Context

Once you can compute derivatives, you use them to describe the real world. The derivative $f'(x)$ **is** the instantaneous rate of change of f with respect to its input. Reading and reporting this rate correctly is a graded skill.

Units matter. The unit of $f'(x)$ is the unit of f divided by the unit of x. If $C(t)$ is a number of acres and t is in weeks, then $C'(t)$ is in acres per week. On the exam, "Using correct units, interpret the meaning of $g'(140)$" wants a full sentence: the *value*, the *quantity*, the *rate word* "per", and the *moment*. For example: "$g'(140) = 2.3$ means that at $x = 140$, the quantity is increasing at about 2.3 units per unit of x."

Straight-Line Motion: Position, Velocity, and Acceleration

For a particle moving on a line, three functions of time are linked by differentiation:



On a velocity-time graph, the area is displacement and the gradient is acceleration

- **position** 位置 $s(t)$;
- **velocity** 速度 $v(t) = s'(t)$ – signed; its sign gives direction;
- **acceleration** 加速度 $a(t) = v'(t) = s''(t)$.

Key readings (frequent exam parts):

- The particle is **at rest** 静止 when $v(t) = 0$.
- It moves **right/up** when $v(t) > 0$ and **left/down** when $v(t) < 0$; it **changes direction** where v changes sign.
- **Speed** 速率 is $|v(t)|$. Speed is *increasing* when v and a have the **same sign** (the particle is speeding up), and *decreasing* when they have **opposite signs**.

Worked example. A particle moves with $s(t) = t^3 - 6t^2 + 9t$. Then $v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$, so it is at rest at $t = 1$ and $t = 3$, and changes direction at each. At $t = 2$, $v(2) = 3(1)(-1) = -3 < 0$ (moving left) and $a(2) = 6(2) - 12 = 0$; just after, $a > 0$ while $v < 0$, so it is **slowing down** there. Distinguish carefully between velocity (has direction) and speed (does not) – the exam tests this exact difference.

A roller coaster is calculus you can feel: velocity is the derivative of position, and acceleration is the derivative of velocity

Rates of Change Beyond Motion

The same derivative idea models any changing quantity: a draining tank, a spreading population, a cooling cup. Whenever a problem says “the rate at which...”, it is describing a derivative. Read the units to know which quantity’s rate you have, then interpret in context.

Introduction to Related Rates

In a **related rates** 相关变化率 problem, several quantities change together over time, and you know some rates but want another. The engine is the chain rule: differentiate a relationship **with respect to time** t. Every variable becomes a function of t, so each derivative picks up a ” / dt” factor. Product and quotient rules may also be needed.

A rising balloon: related rates link how fast radius, volume and height change together

Solving Related Rates Problems

A reliable procedure – and a full-credit template on the exam:

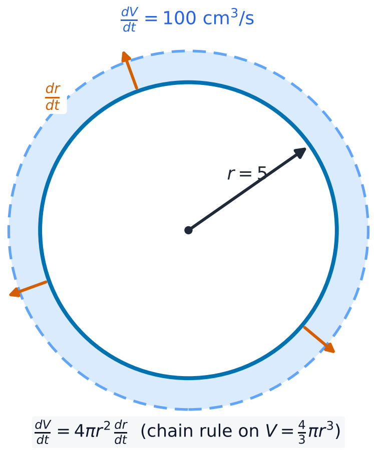
1. **Name the variables** and write down the given rates and the unknown rate (e.g. „$\dfrac{dh}{dt} = -2$ cm/day, find $\dfrac{dV}{dt}$”).
2. **Write an equation** relating the quantities (often a geometric or volume formula).
3. **Differentiate both sides with respect to** t (chain rule) – *before* substituting numbers.
4. **Substitute** the known values at the instant of interest, and solve for the unknown rate.
5. State the answer with **units** and the correct **sign** (a decreasing quantity has a negative rate).

Worked example. Air is pumped into a spherical balloon so its volume grows at $\dfrac{dV}{dt} = 100\ \text{cm}^3/\text{s}$. How fast is the radius growing when $r = 5$ cm? Start from $V = \frac{4}{3}\pi r^3$

and differentiate with respect to t **first**:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \;\Rightarrow\; 100 = 4\pi(5)^2 \frac{dr}{dt} \;\Rightarrow\; \frac{dr}{dt} = \frac{100}{100\pi} = \frac{1}{\pi} \approx 0.32 \text{ cm/s}.$$

Substituting $r = 5$ *too early* is the classic error – differentiate the general relationship, then plug in.



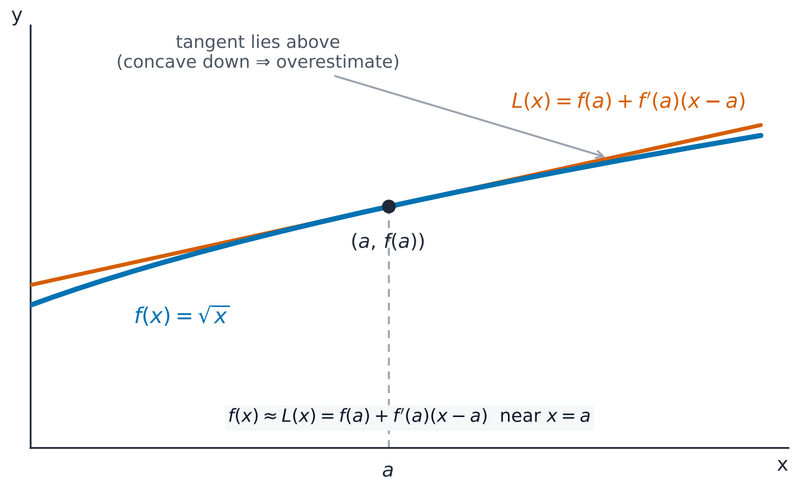
An inflating balloon links dV/dt and dr/dt through the chain rule

Local Linearity and Linearization

Near a point of tangency, a smooth curve looks like its tangent line – this is **local linearity** 局部线性. So the tangent line gives a **linear approximation** 线性近似 (linearization) of the function near that point:

$$f(x) \approx L(x) = f(a) + f'(a)(x-a).$$

Use it to estimate f at an x close to a.



The tangent line is the best linear approximation to the curve near the point of tangency

Worked example. Estimate $\sqrt{4.1}$. Take $f(x) = \sqrt{x}$ and $a = 4$: $f(4) = 2$ and $f'(x) = \dfrac{1}{2\sqrt{x}}$ so $f'(4) = \dfrac{1}{4}$. Then $L(4.1) = 2 + \frac{1}{4}(4.1 - 4) = 2.025$ (the true value is $2.0248\ldots$). Because $\sqrt{x}$ is **concave down**, the tangent lies above the curve, so this is a slight **overestimate** 高估 – justify over/under with the sign of f''.

L'Hospital's Rule for Indeterminate Forms

When direct substitution in a quotient of limits gives the **indeterminate form** 未定式 $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, you may use **L'Hospital's Rule** 洛必达法则:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

provided the right-hand limit exists. **Differentiate the top and bottom separately** (this is *not* the quotient rule), then try the limit again. First confirm the form really is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ – applying the rule to any other form is a mistake.

Worked example. Evaluate $\lim_{x\to 0} \dfrac{\sin x}{x}$. Substituting gives $\frac{0}{0}$, so differentiate top and bottom: $\lim_{x\to 0} \dfrac{\cos x}{1} = \cos 0 = 1$ – confirming the famous limit from Unit 1.

Exam tips

- In motion problems: velocity is the derivative of position, acceleration the derivative of velocity; **speed increases when velocity and acceleration share a sign**.
- For **related rates**, differentiate the relating equation with respect to **time**, then substitute the given values last.
- Use the tangent line for a **linear approximation** near a known point; it is accurate only close by.
- Read the sign of a rate: positive means the quantities move together, negative means opposite.
- Always state units and interpret the answer in context.

Analytical Applications of Differentiation

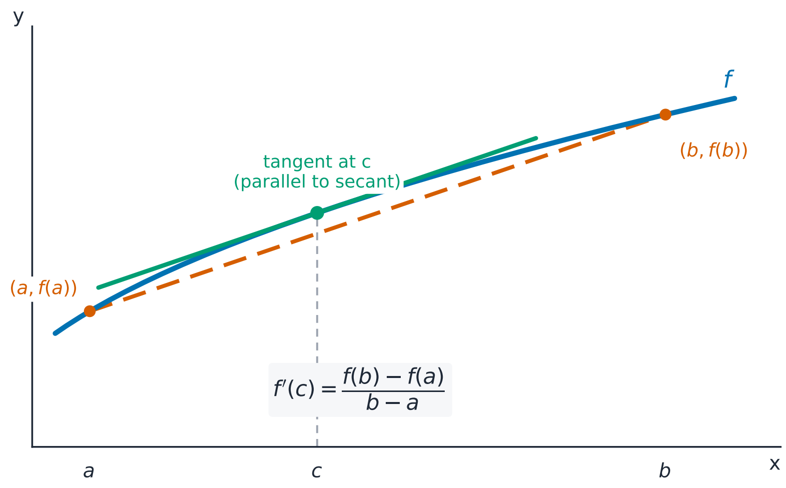
AP Calculus AB

The Mean Value Theorem

The **Mean Value Theorem** 中值定理 (MVT) links the average rate of change to an instantaneous one:

> If f is **continuous** on $[a, b]$ and **differentiable** on (a, b), then there is at least one point c in (a, b) where
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In words: somewhere inside the interval, the instantaneous rate equals the average rate. Geometrically, some tangent line is parallel to the line joining the endpoints.



The Mean Value Theorem: some tangent is parallel to the secant over the interval

Exam skill. Like the IVT, the MVT is an existence theorem, and questions ask you to justify. Full credit needs: (1) state f is continuous on $[a, b]$ and differentiable on (a, b); (2) compute the average rate $\frac{f(b)-f(a)}{b-a}$; (3) conclude "by the MVT there is a c in (a, b) with $f'(c)$ equal to that value." Both hypotheses must be named.

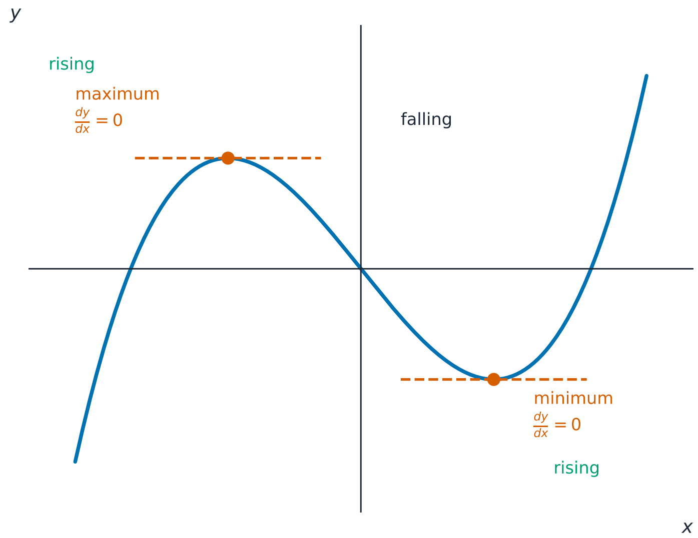
Worked example. Verify the MVT for $f(x) = x^2$ on $[1, 3]$. It is a polynomial, so continuous and differentiable everywhere. The average rate is $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{9 - 1}{2} =$

4. Setting $f'(c) = 2c = 4$ gives $c = 2$, which does lie in $(1, 3)$ – the guaranteed point.

Extreme Values and Critical Points

A mountain peak reflected in a lake: extrema and critical points mark highest and lowest values

The **Extreme Value Theorem** 极值定理 (EVT) guarantees extremes exist: a function **continuous** on a closed interval $[a, b]$ attains **both** an absolute maximum and an absolute minimum on it.



At a maximum or minimum the derivative is zero

A **critical point** 临界点 is an interior point where $f'(x) = 0$ **or** $f'(x)$ does not exist. All **local (relative) extrema** 局部极值 occur at critical points – but **not every critical point is an extremum**. So critical points are the *candidates*; you must test each.

Increasing and Decreasing Intervals

The **first derivative** tells you where f rises or falls:

- $f'(x) > 0$ on an interval $\Rightarrow f$ is **increasing** 递增 there;
- $f'(x) < 0 \Rightarrow f$ is **decreasing** 递减.

On the exam, "find the intervals where f is increasing" means: find the critical points, then test the sign of f' between them, and **justify with the sign of** f' (a stated reason, not just an interval).

The First Derivative Test

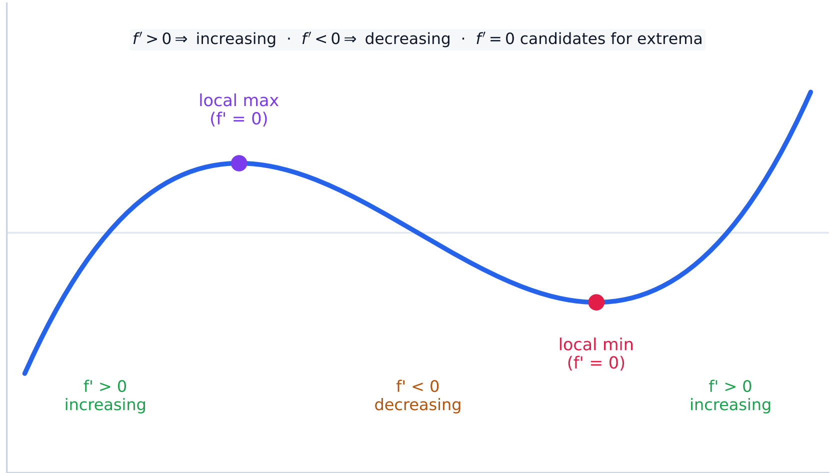
To classify a critical point $x = c$ as a local max, local min, or neither, check how f' changes sign there:

- f' changes $+$ **to** $-$ at $c \Rightarrow$ **local maximum** 极大值;

- f' changes $-$ **to** $+$ at $c \Rightarrow$ **local minimum** 极小值;
- f' does **not** change sign $\Rightarrow$ neither.

Always state the sign change as your justification.

Worked example. Classify the critical points of $f(x) = x^3 - 3x^2$. Here $f'(x) = 3x^2 - 6x = 3x(x-2)$, zero at $x = 0$ and $x = 2$. Testing signs: $f' > 0$ for $x < 0$, $f' < 0$ on $(0, 2)$, and $f' > 0$ for $x > 2$. So f' turns $+ \to -$ at $x = 0$ (a **local maximum**, $f(0) = 0$) and $- \to +$ at $x = 2$ (a **local minimum**, $f(2) = -4$).



Where $f' > 0$ the graph rises and where $f' < 0$ it falls; a local max sits where f' turns $+$ to $-$, a local min where it turns $-$ to $+$.

The Candidates Test for Absolute Extrema

On a **closed interval**, absolute (global) extrema occur only at **critical points or endpoints**. The **candidates test**:

1. List all critical points in $[a, b]$ **and** the two endpoints.
2. Evaluate f at each candidate.
3. The largest output is the absolute maximum; the smallest is the absolute minimum.

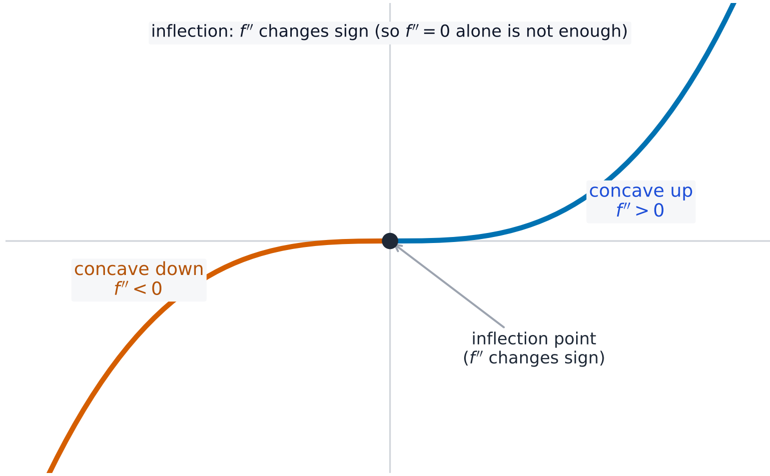
Show the table of values – the comparison is the argument.

Concavity and Points of Inflection

The **second derivative** describes bending:

- $f'' > 0 \Rightarrow f$ is **concave up** 上凹 (curving like a cup; f' is increasing);
- $f'' < 0 \Rightarrow f$ is **concave down** 下凹 (f' is decreasing).

A **point of inflection** 拐点 is where concavity **changes**, i.e. where f'' changes sign (not merely where $f'' = 0$). Report its x-coordinate and justify with the sign change of f''.



Concavity comes from the sign of the second derivative; it flips at an inflection point

The Second Derivative Test

An alternative way to classify a critical point c where $f'(c) = 0$:

- $f''(c) > 0 \Rightarrow$ concave up $\Rightarrow$ **local minimum**;
- $f''(c) < 0 \Rightarrow$ concave down $\Rightarrow$ **local maximum**;
- $f''(c) = 0 \Rightarrow$ the test is **inconclusive** – fall back on the first derivative test.

Special case: if a continuous function has only **one** critical point on an interval and it is a local extremum, that point is also the **absolute** extremum there.

Sketching a Function and Its Derivatives

Key features of f, f', and f'' mirror each other. To sketch or read graphs:

- f increasing $\Leftrightarrow$ f' above the axis; f has a local max $\Leftrightarrow$ f' crosses from $+$ to $-$.
- f concave up $\Leftrightarrow$ f' increasing $\Leftrightarrow$ f'' above the axis; f has an inflection point $\Leftrightarrow$ f' has a local extremum $\Leftrightarrow$ f'' crosses zero.

Connecting f, f-prime, and f-double-prime

This is the skill of reading one graph to describe another. A very common exam setup gives the graph of f' and asks about f: where is f increasing (where $f' > 0$), where are f's extrema (where f' crosses zero, with a sign change), where is f concave up (where f' is increasing). Answer questions about f using the *height and slope* of the f' graph.

Introduction to Optimization

Optimization 最优化 uses the derivative to find the largest or smallest value of a quantity on an interval. It is the candidates/derivative-test machinery applied to a real goal.

Fenced enclosures: optimization finds the dimensions that maximise area for a fixed fence length

Solving Optimization Problems

A dependable procedure:

1. Write the quantity to optimize as a function of **one** variable (use a constraint equation to eliminate extras).
2. State the interval of allowed inputs.
3. Find critical points ($f' = 0$ or undefined) and test them (first- or second-derivative test, or candidates test if the interval is closed).
4. Answer the question asked, **with units and interpretation** in context – the maximum *area*, the minimum *cost*, etc.

Worked example. A farmer has 100 m of fence for a rectangular pen against a wall, so only **three** sides need fencing. Maximize the area. Let the two ends be x and the side parallel to the wall be y; the constraint is $2x + y = 100$, so $y = 100 - 2x$. The area is

$$A(x) = x(100 - 2x) = 100x - 2x^2, \qquad A'(x) = 100 - 4x = 0 \;\Rightarrow\; x = 25.$$

Since $A'' = -4 < 0$ this is a maximum, giving $y = 50$ and $A = 25 \times 50 = 1250$ m^2.

Behaviors of Implicit Relations

All of this extends to **implicitly defined** relations. A critical point of an implicit relation is where $\frac{dy}{dx} = 0$ (horizontal tangent) or is undefined (vertical tangent). Because $\frac{dy}{dx}$ is usually a relation in x **and** y, and the second derivative involves x, y, and $\frac{dy}{dx}$, substitute your first-derivative expression back in when finding $\frac{d^2y}{dx^2}$, then reason about concavity from its sign.

Exam tips

- $f' > 0$ means increasing, $f' < 0$ decreasing; candidates for **extrema** are where $f' = 0$ or is undefined.
- Classify a critical point with the **first-derivative sign change** or the **second-derivative test** ($f'' > 0$ minimum, $f'' < 0$ maximum).
- $f'' > 0$ is concave up, $f'' < 0$ concave down; a **point of inflection** is where concavity changes (f'' changes sign).
- For an **absolute** extremum on a closed interval, also check the endpoints.
- Justify every conclusion by citing the sign of f' or f'' — the exam demands the reasoning, not just the answer.

Integration and Accumulation of Change

AP Calculus AB

Exploring Accumulations of Change

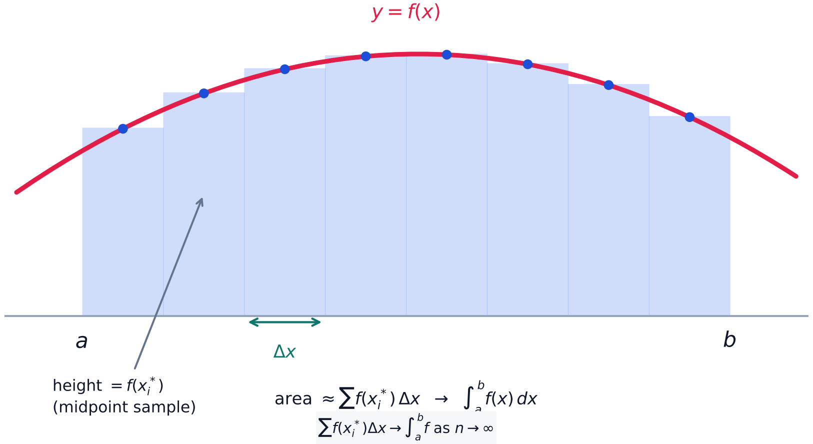
A water tank: integration accumulates change — total volume from a rate of flow

Differentiation found rates. Integration runs the idea in reverse: given a **rate of change**, it finds the **accumulated change** 累积变化. The key picture: the **area** between the graph of a rate function and the x-axis gives the total accumulation.

- If the rate is positive over an interval, the accumulated change is positive; if negative, negative. Area below the axis counts as negative.
- Simple regions (triangles, rectangles) can be found with **geometry** 几何.
- **Units:** the area's unit is the rate's unit times the input's unit. A rate in vehicles-per-hour times hours gives **vehicles**.

Approximating Areas with Riemann Sums

When exact area is hard, approximate it with a **Riemann sum** 黎曼和 – split the interval into subintervals and add up rectangle (or trapezoid) areas. The four standard estimates:



Each rectangle has area $f(x)\,\Delta x$; adding them estimates the area, and as the strips narrow the sum approaches the definite integral.

- **Left** Riemann sum – rectangle height from the **left** endpoint of each subinterval.
- **Right** Riemann sum – height from the **right** endpoint.
- **Midpoint** Riemann sum – height from the **midpoint** 中点.
- **Trapezoidal sum** 梯形法 – average the two endpoint heights (a trapezoid).

Subintervals may be **uniform** (equal width) or **nonuniform** – read widths from the table.

Over- or underestimate? Judge from the behavior of the function: for an **increasing** function, a left sum underestimates and a right sum overestimates; a **trapezoidal** sum overestimates when the function is **concave up** and underestimates when concave down. Exam parts ask you to state which and why.

Worked example. A table gives $f(0) = 3$, $f(2) = 5$, $f(4) = 8$, $f(6) = 9$. Estimate $\int_0^6 f(x)\,dx$ with a **right** Riemann sum of three equal subintervals ($\Delta x = 2$). Use the right endpoint of each strip:

$$2\big(f(2) + f(4) + f(6)\big) = 2(5 + 8 + 9) = 44.$$

Since f is increasing, this right sum is an **overestimate**; the left sum $2(3+5+8) = 32$ would be an underestimate.

Riemann Sums and Integral Notation

As the subinterval widths shrink to zero, the Riemann sum approaches an exact value – the **definite integral** 定积分:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

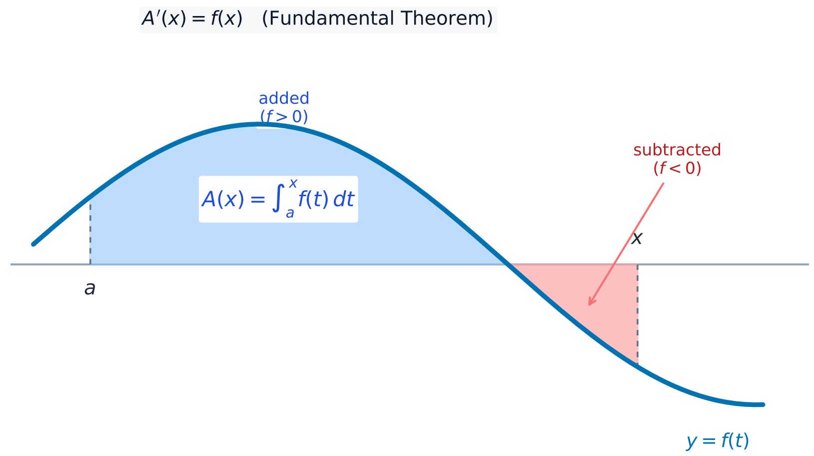
Here Δx_i is the width of the ith subinterval and x_i^* a point inside it. So a definite integral *is* the limit of a Riemann sum, and you should be able to translate each into the other.

The Fundamental Theorem of Calculus and Accumulation Functions

A definite integral with a variable upper limit defines a new **accumulation function** 累积函数. The **Fundamental Theorem of Calculus** 微积分基本定理 (first part) says differentiation undoes this accumulation: if f is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

So if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$ and $g''(x) = f'(x)$. This is the engine behind the very common "let $g(x) = \int_a^x f(t)\,dt$" questions.



The accumulation function adds signed area; the FTC says its derivative is f

Behavior of Accumulation Functions

Because $g'(x) = f(x)$, everything from Unit 5 applies to an accumulation function using the graph of f:

- g is **increasing** where $f > 0$ and **decreasing** where $f < 0$;
- g has a local extremum where f **crosses zero** (with a sign change);
- g is **concave up** where f is **increasing**; inflection points of g occur where f has a local extremum.

To get a *value* of g, compute the signed area: $g(x) = \int_a^x f(t)\,dt$, adding areas above the axis and subtracting areas below.

Properties of Definite Integrals

These properties simplify computation and appear constantly:

$$\int_a^a f = 0, \qquad \int_b^a f = -\int_a^b f, \qquad \int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g,$$
$$\int_a^b k\, f = k \int_a^b f, \qquad \int_a^c f + \int_c^b f = \int_a^b f.$$

The last (splitting at an interior point c) lets you build a total integral from pieces read off a graph.

The Fundamental Theorem and Evaluating Integrals

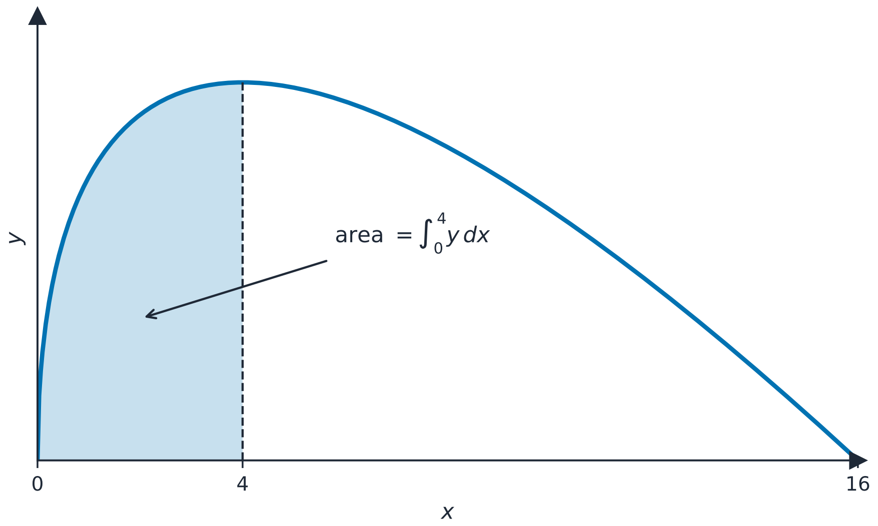
The second part of the Fundamental Theorem evaluates a definite integral using an **antiderivative** 原函数. If $F' = f$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

So: find any antiderivative F, then subtract its values at the two limits. This is how most exact integrals are computed. It also gives the **net change** view: $\int_a^b g'(t)\, dt = g(b) - g(a)$, so a starting value plus accumulated change gives a later value, e.g. $g(5) = g(0) + \int_0^5 g'(t)\, dt$.

Worked example. Evaluate $\int_1^3 (2x+1)\, dx$. An antiderivative is $F(x) = x^2 + x$, so

$$\int_1^3 (2x+1)\, dx = F(3) - F(1) = (9+3) - (1+1) = 12 - 2 = 10.$$



A definite integral is the signed area between the curve and the x-axis

Antiderivatives and Indefinite Integrals

An **indefinite integral** 不定积分 is the family of all antiderivatives, written with a **constant of integration** 积分常数:

$$\int f(x)\, dx = F(x) + C.$$

Reverse each derivative rule to build the basic antiderivatives:

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1), \qquad \int \frac{1}{x}\, dx = \ln|x| + C, \qquad \int e^x\, dx = e^x + C,$$
$$\int \cos x\, dx = \sin x + C, \qquad \int \sin x\, dx = -\cos x + C.$$

Integrating Using Substitution

u-substitution 换元积分法 reverses the chain rule. Choose an inside function $u = g(x)$, so $du = g'(x)\,dx$, and rewrite the integral entirely in u:

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du.$$

Look for a function *and its derivative* both present. For a **definite** integral, either change the limits to u-values or convert back to x before substituting the original limits.

Worked example. Evaluate $\int 2x\cos(x^2)\,dx$. The inside function is $u = x^2$, whose derivative $2x\,dx = du$ is present, so

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

Spotting that $2x$ is exactly $\dfrac{du}{dx}$ is the whole trick.

Long Division and Completing the Square

Two algebraic set-up moves let more integrals fit the basic forms: **polynomial long division** 多项式长除法 when the top degree is $\geq$ the bottom degree of a rational function, and **completing the square** 配方法 to turn a quadratic denominator into a form that integrates to an arctangent or logarithm.

Selecting Techniques for Antidifferentiation

A skill topic: match the integral to a method. Try a basic antiderivative first; look for a u-substitution (an inside function whose derivative is also present); use algebra (division, completing the square, splitting a fraction) to reshape the integrand into a standard form. Naming the structure first prevents wasted effort.

Exam tips

- Integration is antidifferentiation; use the **power rule** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and don't forget the $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f'(x)\,dx = f(b) - f(a)$, and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.

- A definite integral is a signed area (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.

Differential Equations

AP Calculus AB

Modeling Situations with Differential Equations

A **differential equation** 微分方程 is an equation that relates a function to its own derivatives. It describes a situation by its *rate of change*. For example, "the rate of change of a quantity is proportional to its size" becomes

$$\frac{dy}{dt} = ky.$$

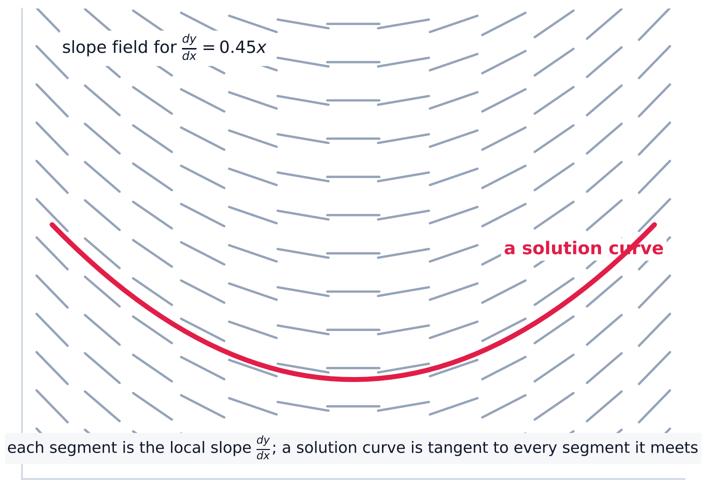
Learning to translate a sentence about a rate into a differential equation is the first skill of this unit.

Verifying Solutions for Differential Equations

A **solution** 解 of a differential equation is a function that makes it true. You can **verify** a proposed solution by differentiating it and substituting into the equation: if both sides match, it is a solution. Note a differential equation usually has **infinitely many** solutions – a whole family – differing by a constant.

Sketching Slope Fields

A **slope field** 斜率场 draws the differential equation as a grid of short segments: at each point (x, y) the segment has slope $\dfrac{dy}{dx}$ evaluated there. To sketch one, plug several points into the right-hand side and draw a small segment with that slope at each. The picture shows the shape of the solution curves without solving.



Each short segment has the slope $\dfrac{dy}{dx}$ at that point; a solution curve threads through, staying tangent to the field.

Exam skill. A common part gives a portion of a slope field and asks you to sketch the particular solution through a given point – start at the point and follow the segments, staying tangent to them.

Reasoning Using Slope Fields

Solutions to a differential equation are **functions or families of functions**. Read a slope field to reason about behavior: where the segments are flat ($\frac{dy}{dx} = 0$) the solution is momentarily level; where they steepen the solution rises or falls faster; horizontal rows of equal slope suggest the rate depends only on y (or only on x).

Finding General Solutions Using Separation of Variables

Many exam differential equations are solved by **separation of variables** 分离变量法. If $\frac{dy}{dx}$ factors into a function of x times a function of y, move all y's to one side and all x's to the other, then integrate both sides:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x)\, dx.$$

Add the constant of integration once (on the x-side). This gives the **general solution** 通解.

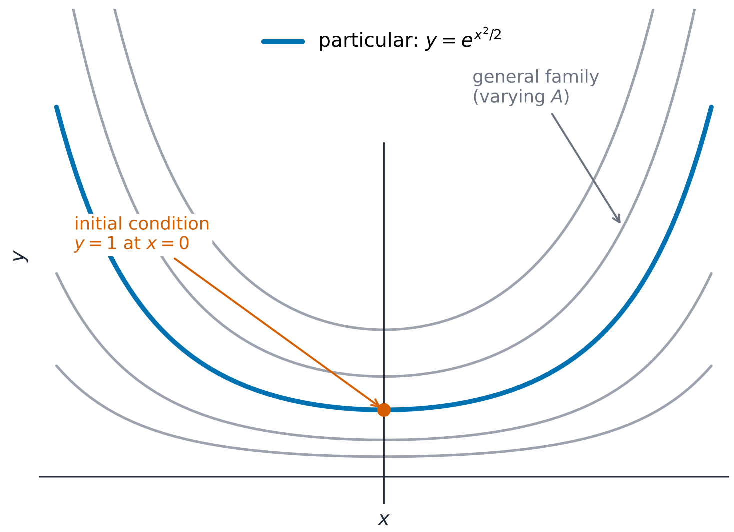
Worked example. Solve $\frac{dy}{dx} = xy$ with the initial condition $y(0) = 2$. Separate and integrate:

$$\int \frac{dy}{y} = \int x\, dx \;\Rightarrow\; \ln|y| = \frac{x^2}{2} + C \;\Rightarrow\; y = Ae^{x^2/2}.$$

Applying $y(0) = 2$ gives $A = 2$, so the particular solution is $y = 2e^{x^2/2}$.

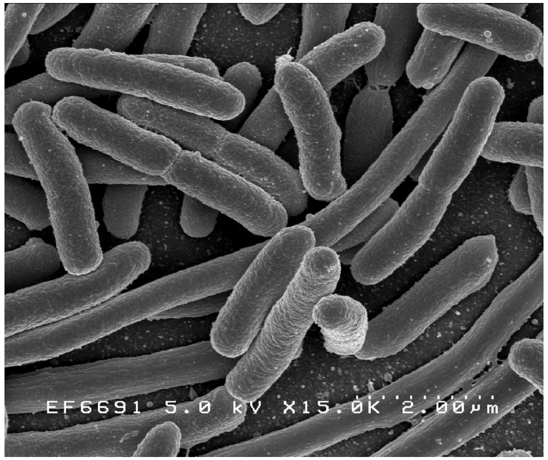
Finding Particular Solutions Using Initial Conditions and Separation of Variables

An **initial condition** 初始条件 – a known point (x_0, y_0) – pins down one curve from the family. Substitute it to solve for the constant C; the result is the **particular solution** 特解. There is exactly one solution through a given point. Watch for **domain restrictions** 定义域限制: keep the branch that contains the initial point (for example, the correct sign of a square root).



The constant gives a family of curves; an initial condition picks out one

Exponential Models with Differential Equations



E. coli under a microscope: differential equations model exponential growth of populations

The most important model is **exponential growth and decay 指数增长与衰减**. The equation $\dfrac{dy}{dt} = ky$ (rate proportional to size) has the solution:

$$y = y_0 \, e^{kt},$$

where y_0 is the value at $t = 0$. Here $k > 0$ gives growth and $k < 0$ gives decay. You can derive this by separation of variables ($\int \frac{dy}{y} = \int k \, dt$), and it models processes like population growth, radioactive decay, and Newton's law of cooling. A follow-up part may ask for $\dfrac{d^2y}{dt^2}$: differentiate $\dfrac{dy}{dt} = ky$ again (using $\dfrac{dy}{dt} = ky$) to express it in terms of y.

Worked example. A sample decays by $\dfrac{dy}{dt} = -0.10\, y$ (in years) from $y_0 = 50$ g. The solution is $y = 50e^{-0.10t}$, so after 10 years $y = 50e^{-1} = 18.4$ g. Its **half-life** solves $25 = 50e^{-0.10t}$, i.e. $e^{-0.10t} = \frac{1}{2}$, giving $t = \dfrac{\ln 2}{0.10} = 6.9$ years – independent of the starting amount.

A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope — you must integrate to recover the function.

Applications of Integration

AP Calculus AB

Average Value of a Function

The **average value** 平均值 of a continuous function f over $[a, b]$ is the integral divided by the interval length:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It is the constant height of a rectangle on $[a, b]$ with the same area as under f. Do not confuse it with the **average rate of change** (which divides *change in* f by the interval), or with a Riemann-sum average of a few values – the exam distinguishes these carefully. Report units in context (e.g. the average rate "in vehicles per hour").

Worked example. The average value of $f(x) = x^2$ on $[0, 3]$ is

$$f_{\text{avg}} = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}(9) = 3.$$

Position, Velocity, and Acceleration Using Integrals

Integration reverses the motion links of Unit 4. For a particle in straight-line motion over $[t_1, t_2]$:

- **Displacement** 位移 (net change in position) $= \displaystyle\int_{t_1}^{t_2} v(t)\,dt$.
- **Total distance travelled** 总路程 $= \displaystyle\int_{t_1}^{t_2} |v(t)|\,dt$ – integrate **speed**, so direction changes add up instead of cancelling.
- **Position** at a later time $= x(t_1) + \displaystyle\int_{t_1}^{t} v(s)\,ds$; **velocity** from acceleration $= v(t_1) + \int a$.

The displacement-versus-distance distinction (with the absolute value) is a classic graded point.

Accumulation Functions and Definite Integrals in Applied Contexts

A function defined as an integral accumulates a rate of change. The **net change** 净变化 theorem is the everyday tool: the definite integral of a rate over an interval gives the net change of the quantity:

$$(\text{final}) = (\text{initial}) + \int_a^b (\text{rate})\,dt.$$

So "how much water is in the tank at $t = 5$" = starting amount + $\int_0^5(\text{inflow} - \text{outflow})\,dt$. Watch signs (in minus out) and units. Many multi-part FRQs are built entirely on this idea – often paired with "write, but do not evaluate, an integral expression that gives the total...".

Finding the Area Between Curves (Functions of x)

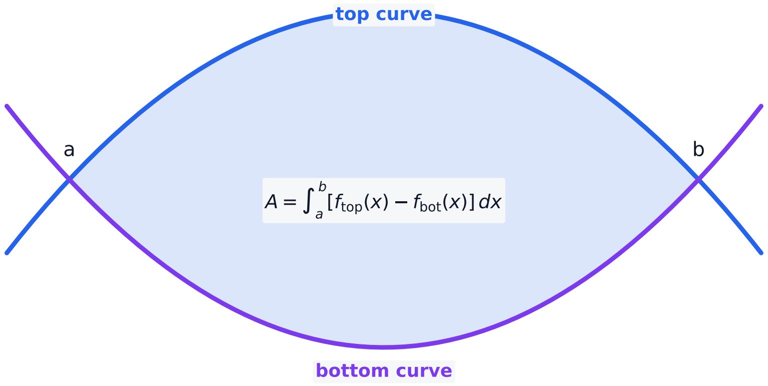
The area between two curves $y = f(x)$ (top) and $y = g(x)$ (bottom) on $[a, b]$ is:

$$A = \int_a^b \left[f(x) - g(x)\right] dx = \int_a^b (\text{top} - \text{bottom})\,dx.$$

Find the intersection points first to set the limits, and always subtract bottom from top.

Worked example. Find the area between $y = x$ and $y = x^2$. They cross at $x = 0$ and $x = 1$, and on $(0, 1)$ the line $y = x$ is on top, so

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$



The shaded region runs from one intersection a to the next b; its area is $\int_a^b (top - bottom)\,dx$.

Finding the Area Between Curves (Functions of y)

Some regions are easier described with **horizontal** slices – integrate with respect to y:

$$A = \int_c^d \left[x_{\text{right}}(y) - x_{\text{left}}(y)\right] dy = \int_c^d (\text{right} - \text{left})\,dy.$$

Choose x- or y-slices to keep a single top/bottom (or right/left) pair over the whole region.

Area Between Curves With More Than Two Intersections

If the curves cross more than twice, one function is on top for part of the region and the other on top elsewhere. Split into a **sum of integrals** at each crossing, or integrate the **absolute value of the difference**:

$$A = \int_a^b \left|f(x) - g(x)\right| dx.$$

Determine which curve is higher on each subinterval before writing the integrals.

Volumes with Cross Sections: Squares and Rectangles

A potter shaping a vessel: solids of revolution are volumes formed by rotating a plane region

If a solid has a known **cross section** 横截面 perpendicular to an axis, its volume is the integral of the cross-sectional **area**:

$$V = \int_a^b A(x)\,dx.$$

For **square** cross sections with side equal to the region's height $s(x) = f(x) - g(x)$, use $A(x) = [s(x)]^2$; for a **rectangle** of height $k \cdot s(x)$, use $A(x) = k\,[s(x)]^2$. The exam phrase "the base of a solid... cross sections are squares/rectangles" signals exactly this.

Volumes with Cross Sections: Triangles and Semicircles

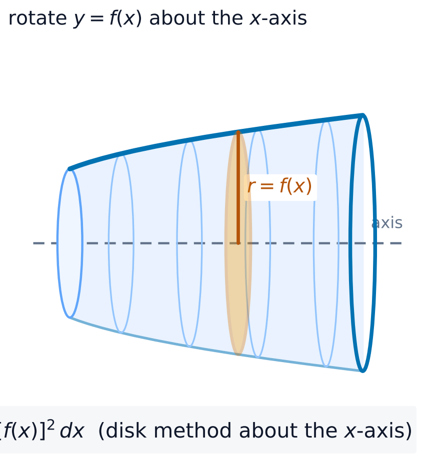
Same method, different area formula: for an equilateral **triangle** 三角形 of side s, $A = \frac{\sqrt{3}}{4}s^2$; for a **semicircle** 半圆 of diameter s, $A = \frac{\pi}{8}s^2$. Set the shape's key length equal to the region's slice length $s(x)$, then integrate $A(x)$.

Volume with the Disc Method (About the x- or y-Axis)

Revolving a region around an axis makes a **solid of revolution** 旋转体. If the region touches the axis, each slice is a **disc** 圆盘 of radius $r =$ the function value:

$$V = \pi \int_a^b [r(x)]^2\, dx.$$

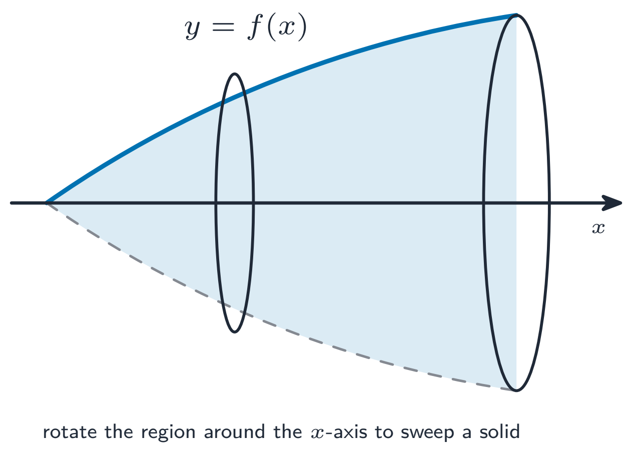
Around the x-axis, $r = f(x)$ and integrate in x; around the y-axis, express x as a function of y and integrate in y.



The disc method: rotating y=f(x) about the axis sweeps out disks of radius f(x)

Worked example. Revolve the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ about the x-axis. Each disc has radius $r = \sqrt{x}$, so

$$V = \pi \int_0^4 (\sqrt{x})^2\, dx = \pi \int_0^4 x\, dx = \pi \left[\frac{x^2}{2}\right]_0^4 = 8\pi.$$



Rotating a region about an axis sweeps out a solid of revolution

Volume with the Disc Method (About Other Axes)

Around any horizontal or vertical line $y = k$ or $x = k$, the radius is the **distance** from the curve to that line, e.g. $r(x) = |f(x) - k|$. Set up the radius carefully, then use $V = \pi \int r^2$.

Volume with the Washer Method

When the region does **not** touch the axis, each slice is a **washer** 垫圈 (a ring) with an outer radius R and inner radius r:

$$V = \pi \int_a^b \left([R]^2 - [r]^2\right) dx.$$

R is the distance from the axis to the **farther** boundary and r to the **nearer** one. Around a line other than an axis, both radii are measured as distances to that line – a very common exam variation (e.g. revolving about $y = -2$).

Exam tips

- Area between curves is $\int(\text{top} - \text{bottom})\,dx$ — find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f\,dx$.
- Accumulated change is $\int$ of a rate: total = initial value + $\int_a^b(\text{rate})\,dt$.
- Integration means "adding up infinitely many tiny pieces" — set up the integrand as one thin slice.