Course Companion

AP Calculus AB

Every topic handout in one volume

全部专题讲义合集

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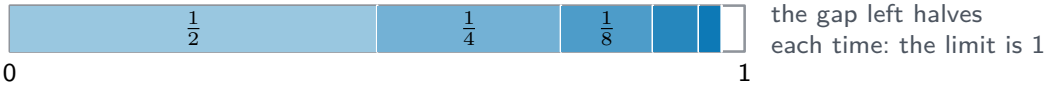
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Limits and Continuity

AP Calculus AB

Introducing Calculus: Can Change Occur at an Instant?

Why it matters. About 2,400 years ago the Greek thinker Zeno argued that a runner can never finish a race: first she must cover half the distance, then half of what is left, then half of that, for ever. Infinitely many steps, he said, cannot be finished. The answer took more than two thousand years: the distances $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ get as close to 1 as we like, so their limit is exactly 1. Newton and Leibniz built calculus on this idea in the 1660s-1680s, and Augustin-Louis Cauchy gave it a careful definition in 1821. Every derivative and integral in this course is a limit.



Zeno's halves add up to a limit of 1.

Calculus is the mathematics of **change** 变化 and of **accumulation** 累积. It answers two big questions: how fast is something changing *right now*, and how much has piled up *so far*? Unit 1 builds the one tool both questions rest on – the **limit** 极限.

Start with a puzzle. A car's speedometer reads 60 km/h. What does that mean *at a single instant* 瞬间? Speed is distance over time. But at one instant no time passes and no distance is covered, so the fraction looks like $\frac{0}{0}$ – undefined.

- The **average rate of change** 平均变化率 uses a whole *interval* 区间: the change in one quantity divided by the change in another. It divides by zero, and so is undefined, when the change in the input would be zero.
- The **instantaneous rate of change** 瞬时变化率 is what we want *at a point*. It is the value the average rate **approaches** 趋近 as the interval shrinks toward zero length.

The clever move is not to plug in zero (undefined), but to watch what the average rate *approaches* as the interval gets smaller and smaller. That approaching value is a **limit**. So calculus lets us describe change at an instant – as a limit of average rates over ever-shorter intervals. This one idea powers the **derivative** 导数 (Unit 2) and, run in reverse, the **integral** 积分 (Unit 6). Everything else in this unit defines limits carefully and computes them reliably.

Defining Limits and Using Limit Notation

Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c – **but not equal to** c. We write

$$\lim_{x \to c} f(x) = R$$

and read it: "the limit of $f(x)$, as x approaches c, equals R."



A smooth bridge curve: limits describe the value a graph approaches as we zoom in

The last words are the heart of a limit: it describes the **behavior** 行为 of f *near* c, not the value *at* c. The function may be undefined at c, or defined but equal to something else – the limit does not care.

A limit can be shown in three ways: **graphically** 用图象, **numerically** 用数值 (a table), and **analytically** 用解析式 (algebra). Learning to move between these representations is a core skill.

*(Note: the epsilon-delta definition of a limit is **not** tested on the AP Exam, so this handout does not use it.)*

Estimating Limit Values from Graphs

A graph is often the fastest way to read a limit. To find $\lim_{x \to c} f(x)$, run your finger along the curve toward $x = c$ from each side and ask: what height is the curve heading for?

- Trace from the **left** (inputs smaller than c): this gives the **left-hand limit** 左极限, $\lim_{x \to c^-} f(x)$.
- Trace from the **right** (inputs larger than c): this gives the **right-hand limit** 右极限, $\lim_{x \to c^+} f(x)$.

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- These are the **one-sided limits** 单侧极限. If both head to the *same* height R, then the two-sided limit exists and $\lim_{x\to c} f(x) = R$.

Crucially, **ignore the point itself**. Graphs mark the difference between the limit and the value:

- An **open circle** 空心圆 marks a height the curve *approaches* but does not reach – a "hole" 空洞.
- A **closed circle** 实心圆 marks the actual value $f(c)$.

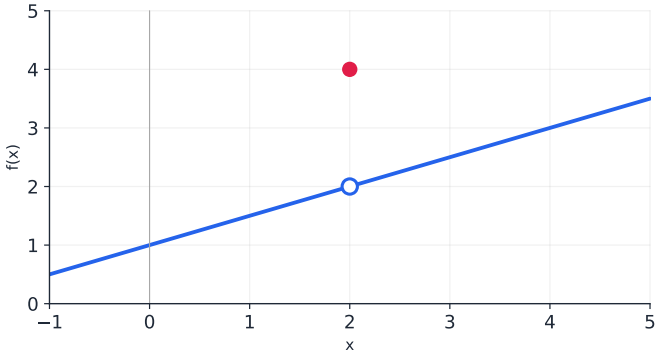
A limit **does not exist** (often written **DNE**) when the two sides disagree (a **jump** 跳跃), when the function is **unbounded** 无界 (grows without limit), or when it **oscillates** 振荡 near c with fluctuations that do not shrink toward one value. Oscillation alone does not prevent a limit: $x\sin(1/x) \to 0$ as $x \to 0$. For example:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x\to 0} \frac{|x|}{x} \text{ DNE}, \qquad \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \text{ DNE}.$$

Watch the **scale** 比例 of a graph: a zoomed-out picture can hide important behavior near a point, so confirm with algebra when you can.

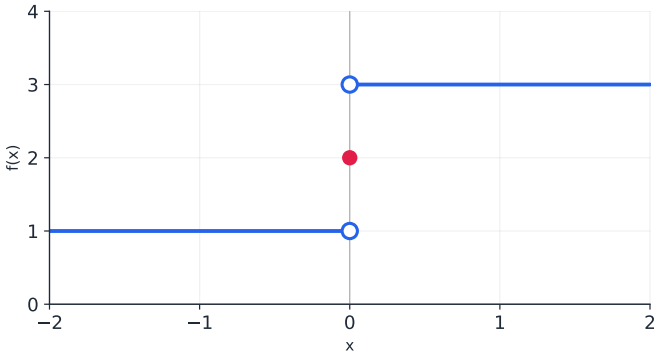
■ Worked comparison: nearby values and one point

In this teacher-created graph, read the approaching heights before the filled point.



A curve with an open point at (2,2) and a filled point at (2,4).

Both one-sided limits at 2 equal 2, so the two-sided limit is 2. The filled point gives $f(2) = 4$. This is a removable discontinuity: set $f(2) = 2$ to make the function continuous.



One-sided heights 1 and 3 at 0, with a separate filled point at height 2.

Here the left-hand limit at 0 is 1, the right-hand limit is 3, and $f(0) = 2$. The unequal one-sided limits mean the two-sided limit does not exist. No choice of $f(0)$ changes the nearby branches; the jump cannot be removed by changing one point.

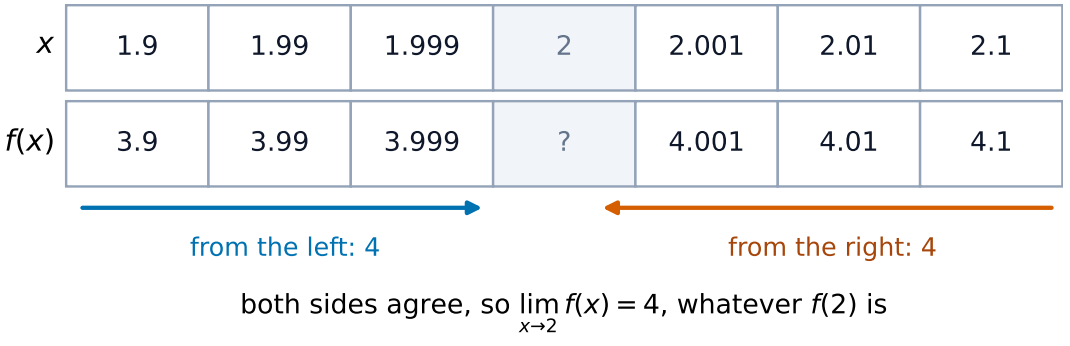
Estimating Limit Values from Tables

When you have data or a formula but no picture, a **table** 表格 of values estimates a limit numerically. Choose inputs that creep toward c from both sides and watch the outputs.

For example, to estimate $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ (which is $\frac{0}{0}$ at $x = 2$):

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4, so we estimate the limit is 4. A table only *suggests* a value – it is a numerical estimate, not a proof.



Values from both sides close in on 4, so the limit is 4 whatever f(2) is

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Determining Limits Using Algebraic Properties of Limits

Most limits are found **analytically** using **limit theorems** 极限定理. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, the limit of a combination is the same combination of the limits:

- **Sum / difference:** $\lim\limits_{x\to c}\big[f(x)\pm g(x)\big] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$
- **Product:** $\lim\limits_{x\to c}\big[f(x)\,g(x)\big] = \lim\limits_{x\to c} f(x)\cdot \lim\limits_{x\to c} g(x)$
- **Quotient:** $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided the bottom limit is not 0.
- **Composite** 复合函数: if g is continuous at $\lim_{x\to c} f(x)$, then $\lim\limits_{x\to c} g\big(f(x)\big) = g\Big(\lim\limits_{x\to c} f(x)\Big)$.

The practical rule: for a function built from polynomials, roots, and the like, first try **direct substitution** 直接代入 – put $x = c$ in. If you get a real number, that is the limit. One-sided limits obey the same theorems, read from one direction only.

Determining Limits Using Algebraic Manipulation

Direct substitution sometimes gives the **indeterminate form** 未定式 $\frac{0}{0}$. This does **not** mean the limit fails – it means you must rewrite the function into an **equivalent form** 等价形式 that removes the trouble, then substitute. Three standard moves:

- **Factor and cancel** 因式分解并约分 a **rational function** 有理函数. Example: $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$.
- Multiply by the **conjugate** 共轭 to simplify a **radical** 根式. Example: $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x} = \lim\limits_{x\to 0}\dfrac{x}{x\big(\sqrt{x+1}+1\big)} = \dfrac{1}{2}$.
- **Use alternate forms** of trigonometric functions (identities) to simplify.

The cancelled factor is why the original graph had a hole: the two functions agree everywhere except at $x = c$, so they share the same limit there.

Selecting Procedures for Determining Limits

This is a skill topic, not new content: choose the right tool for the limit in front of you.

1. **Try direct substitution first.** A real answer means you are done.
2. Getting $\frac{0}{0}$? **Rewrite** – factor and cancel, or use the conjugate, or a trig identity – then substitute.
3. A **non-zero number over** 0 (like $\frac{5}{0}$)? The limit is infinite or DNE – check the sign from each side (see vertical asymptotes below).
4. As $x \to \pm\infty$? Compare the **dominant** 主导 terms (see limits at infinity).

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5. Trapped between two functions? The **squeeze theorem** may apply.

Determining Limits Using the Squeeze Theorem

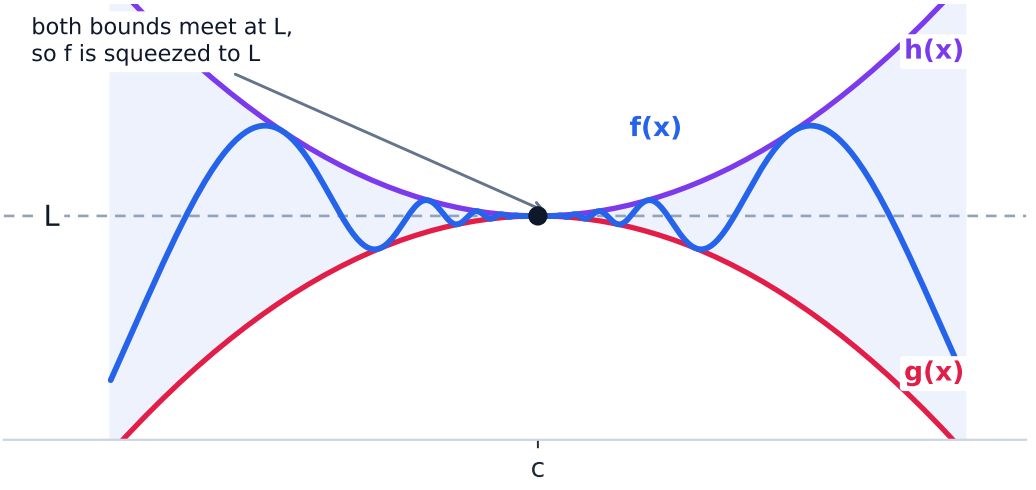
The **squeeze theorem** 夹逼定理 (also called the sandwich theorem) finds a limit by trapping the function between two others. If $g(x) \le f(x) \le h(x)$ near c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then f is squeezed to the same place: $\lim_{x \to c} f(x) = L$.

The two famous results proved this way, both used throughout calculus, are:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$



if $g \le f \le h$ near c and $\lim g = \lim h = L$, then $\lim f = L$

The wiggly f is trapped between g and h. Because both bounds meet at L, f has nowhere to go but L.

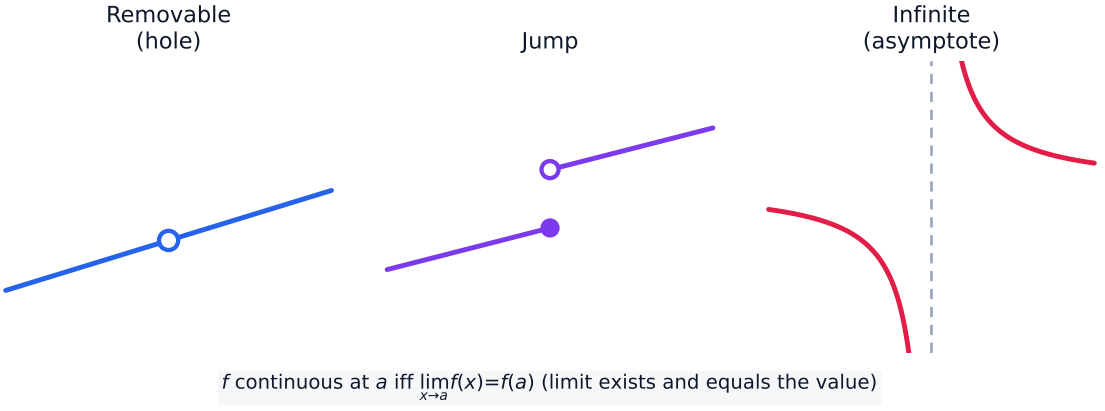
Connecting Multiple Representations of Limits

Another skill topic: the same limit lives in a graph, a table, and an algebraic form, and you should be able to translate between them. A graph shows the shape and any holes or jumps; a table gives numerical evidence; algebra gives an exact value and a reason. Strong answers use one representation to *confirm* another.

Exploring Types of Discontinuities

A function is **discontinuous** 间断 at c when its graph "breaks" there. Three common types are:

- **Removable discontinuity** 可去间断 – a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** 跳跃间断 – the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** 无穷间断 – the function blows up to $\pm\infty$ at a **vertical asymptote** 垂直渐近线.



Three ways a graph can break: a removable hole, a jump, and an infinite (asymptote) discontinuity.

■ Worked comparison: classify and justify a repair

At 2, $A(x) = (x^2 - 4)/(x - 2)$ simplifies to $x + 2$ for $x \neq 2$. Its limit is 4; define $A(2) = 4$ to remove the hole. In contrast, $B(x) = 1$ for $x < 2$ and 3 for $x \geq 2$ has unequal finite one-sided limits 1 and 3. Changing $B(2)$ cannot remove its jump. For $C(x) = 1/(x - 2)$, the left side tends to $-\infty$ and the right to $+\infty$: an infinite discontinuity, with no finite limit to use as a repair.

Defining Continuity at a Point

Continuity is defined by a three-part test. A function f is **continuous** 连续 at $x = c$ exactly when all three hold:

$$f(c) \text{ exists} \quad \text{and} \quad \lim_{x \to c} f(x) \text{ exists} \quad \text{and} \quad \lim_{x \to c} f(x) = f(c)$$

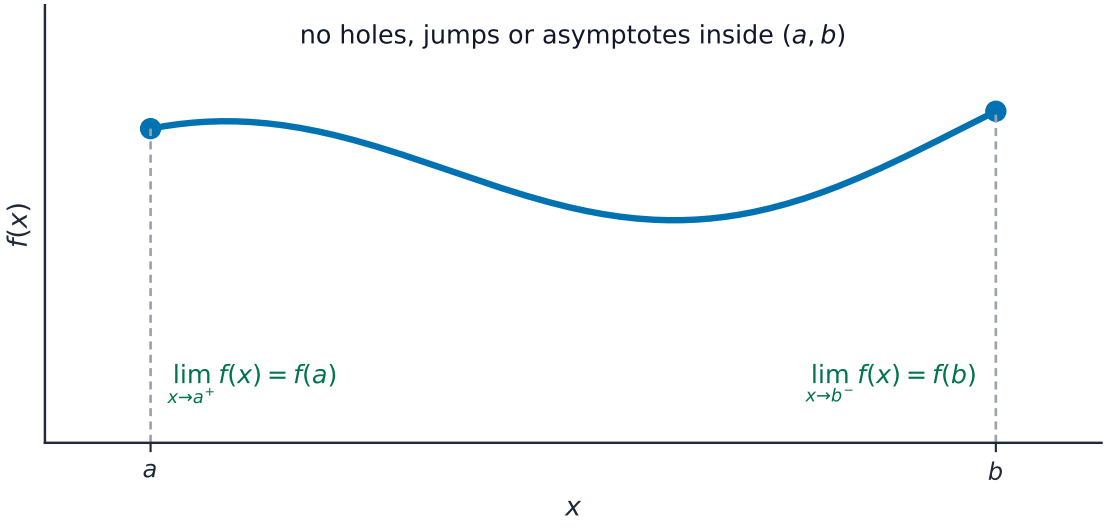
In words: the point is there, the limit is there, and the two agree. If any one fails, f is discontinuous at c. This test is the backbone of nearly every continuity question, so learn it as a checklist.

Confirming Continuity over an Interval

A function is **continuous on an interval** 在区间上连续 if it is continuous at *every* point of that interval. You rarely check point by point, because whole families are continuous on their domains:

> **Polynomial, rational, power, exponential 指数, logarithmic 对数, and trigonometric 三角 functions are continuous at every point of their domains.**

So a rational function is continuous everywhere *except* where its denominator is zero; $\ln x$ is continuous for $x > 0$; and so on. Knowing this lets you declare continuity quickly and correctly.



Continuous on [a, b]: no breaks inside and the one-sided limits match at the ends

Removing Discontinuities

If the **limit exists** at a hole, the discontinuity is **removable**: redefine the function at that one point to equal the limit, and the graph is repaired. Formally, set the missing value to $\lim_{x \to c} f(x)$.

For a **piecewise-defined function** 分段函数, continuity at a boundary $x = c$ needs the two pieces to *meet*: the left piece's value, the right piece's value, and $f(c)$ must all be equal. This is a common exam setup – you solve for a **parameter** 参数 (an unknown constant) that makes the pieces match:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

> **Worked example.** Find the constant k that makes this function continuous at $x = 2$:

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$$f(x) = \begin{cases} kx + 1, & x < 2 \\ x^2 - k, & x \geq 2 \end{cases}$$

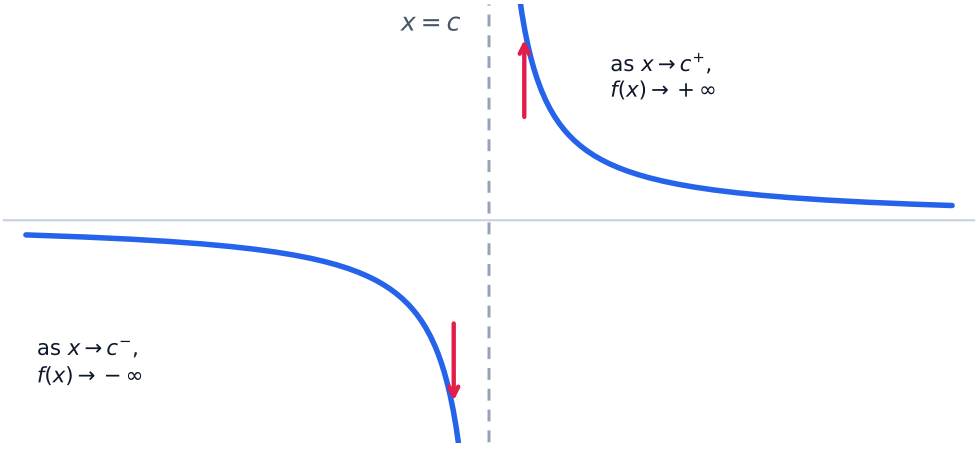
- Left-hand limit: $\lim_{x \to 2^-} (kx + 1) = 2k + 1$.
- Right-hand limit and value: $\lim_{x \to 2^+} (x^2 - k) = f(2) = 4 - k$.
- Set them equal: $2k + 1 = 4 - k$, so $3k = 3$ and $k = 1$.

Check: with $k = 1$ both pieces give 3 at $x = 2$, so all three conditions of the continuity test hold.

Connecting Infinite Limits and Vertical Asymptotes

The idea of a limit extends to **infinite limits** 无穷极限. When a function grows without bound near $x = c$, we write $\lim_{x \to c} f(x) = \pm\infty$. This describes a **vertical asymptote** at $x = c$: the graph hugs the vertical line $x = c$ and shoots off toward $\pm\infty$.

This happens where a **non-zero number is divided by something approaching** 0, such as at a zero of a denominator that does not cancel. Always check each side separately – the two sides can shoot **opposite** ways (one to $+\infty$, one to $-\infty$).



vertical asymptote $x = c$ when $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$

Near a vertical asymptote the graph hugs the line $x = c$. Here the left side falls to $-\infty$ and the right side climbs to $+\infty$, so the two-sided limit does not exist.

Worked example. Find $\lim_{x \to 1^-} \dfrac{x + 2}{x - 1}$ and $\lim_{x \to 1^+} \dfrac{x + 2}{x - 1}$.

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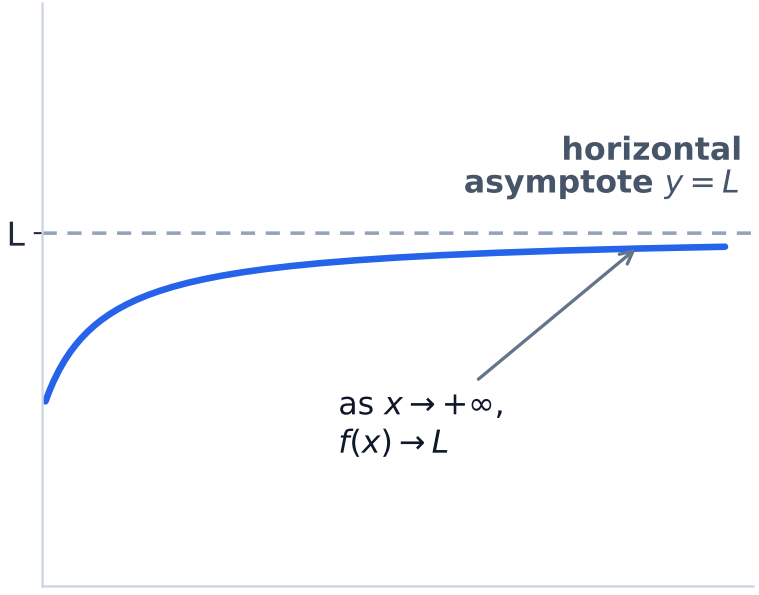
Substitution gives $\frac{3}{0}$: a non-zero number over 0, so each one-sided limit is infinite. Decide the sign from each side:

- For x just below 1 (try $x = 0.99$): top ≈ 3, bottom $= -0.01$, a small negative number. So $\lim\limits_{x\to 1^-} \dfrac{x+2}{x-1} = -\infty$.
- For x just above 1 (try $x = 1.01$): top ≈ 3, bottom $= +0.01$. So $\lim\limits_{x\to 1^+} \dfrac{x+2}{x-1} = +\infty$.

The line $x = 1$ is a vertical asymptote, and the two-sided limit does not exist.

Connecting Limits at Infinity and Horizontal Asymptotes

We can also let the **input** grow: **limits at infinity** 无穷远处的极限 describe the **end behavior** 末端行为 of a function as $x \to \pm\infty$. If the outputs settle toward a finite value L, then $y = L$ is a **horizontal asymptote** 水平渐近线.



As $x \to +\infty$ the curve flattens toward the line $y = L$. The horizontal asymptote records this end behavior.

For a rational function, compare the **degrees** 次数 of the top and bottom:

- top degree $<$ bottom degree $\Rightarrow$ limit is 0 (asymptote $y = 0$);
- top degree $=$ bottom degree $\Rightarrow$ limit is the **ratio of the leading coefficients** 首项系数之比;
- top degree $>$ bottom degree $\Rightarrow$ the function is unbounded (no horizontal asymptote).

Worked example. Find $\lim\limits_{x\to\infty} \dfrac{3x^2 - 5x}{6x^2 + 1}$.

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The degrees are equal (2 and 2), so the limit is the ratio of the leading coefficients. To show why, divide every term by x^2:

$$\lim_{x\to\infty}\frac{3x^2-5x}{6x^2+1}=\lim_{x\to\infty}\frac{3-\frac{5}{x}}{6+\frac{1}{x^2}}=\frac{3-0}{6+0}=\frac{1}{2}.$$

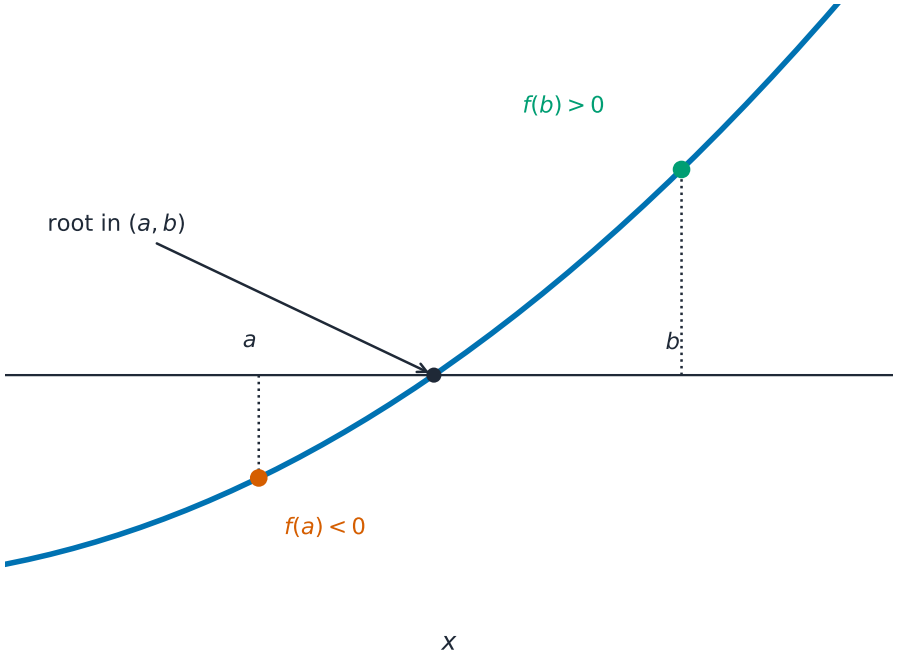
So $y=\frac{1}{2}$ is a horizontal asymptote.

More generally, we compare the **relative magnitudes** 相对大小 (relative growth rates) of functions: far out, an exponential beats any polynomial, and a polynomial beats any logarithm. On the exam, "as $t\to\infty$, which quantity is larger/where does the rate settle?" is answered with a limit at infinity.

Working with the Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem** 介值定理 is an **existence theorem** 存在性定理 – it guarantees a value exists without telling you where:

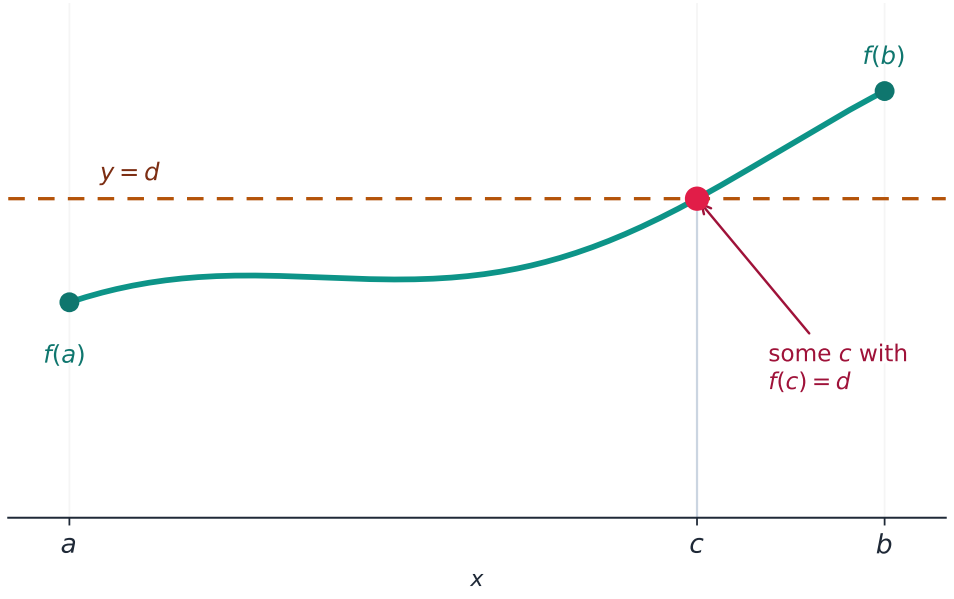
> If f is **continuous** on the closed interval $[a,b]$, and d is any number **between** $f(a)$ and $f(b)$, then there is **at least one** number c in (a,b) with $f(c)=d$.



Opposite signs of f(a) and f(b) trap a root between a and b

An unbroken curve cannot skip a height between its endpoints – it must pass through every one.

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A continuous curve from $(a, f(a))$ to $(b, f(b))$ must cross every height d in between at least once.

Exam skill – how to justify with the IVT. These questions appear almost every year (for example, "Must there be a value c with $R(c) = 155$?" or "Is there a time when $r'(t) = -6$?"). A full-credit justification has three moves:

1. **State continuity.** Say the function is continuous on $[a, b]$ (often because it is differentiable, or given continuous).
2. **Show d is trapped.** Compute the two endpoint values and show the target d lies **between** them, e.g. $f(a) < d < f(b)$.
3. **Conclude by name.** "By the Intermediate Value Theorem, there is a c in (a, b) with $f(c) = d$."

Skipping the continuity statement, or not showing d is between the endpoints, loses the point – the theorem *requires* both conditions.

> **Worked example.** Show that $f(x) = x^3 - x - 1$ has a zero between $x = 1$ and $x = 2$.

1. f is a polynomial, so it is continuous on $[1, 2]$.
2. $f(1) = 1 - 1 - 1 = -1$ and $f(2) = 8 - 2 - 1 = 5$, so $f(1) < 0 < f(2)$.
3. By the Intermediate Value Theorem, there is a c in $(1, 2)$ with $f(c) = 0$.

The theorem says nothing about *where* c is or how many such values there are (here $c \approx 1.32$, found with a calculator).

Common mistakes with the IVT.

- Using it on a function that is not continuous on the whole interval: $g(x) = \frac{1}{x}$ has $g(-1) = -1$ and $g(1) = 1$, but g is never 0, because g breaks at $x = 0$.

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- Reading it backwards: a continuous f can reach a height *outside* $[f(a), f(b)]$ too, and can reach d more than once. The theorem only promises *at least one* c for each d between the endpoint values.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ – the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes at denominator zeros that remain after common factors are cancelled; check the one-sided limits.

Common mistakes.

- Saying a limit does not exist because $f(a)$ is undefined. A hole in the graph does not stop the limit from existing.
- Stopping at $\frac{0}{0}$ and writing "0" or "undefined". $\frac{0}{0}$ is a signal to factor, rationalise or simplify, not an answer.
- Using the IVT without stating that the function is **continuous** on the closed interval - on the exam, the justification earns no credit without it.
- Writing $\lim_{x\to a} f(x) = \infty$ and then calling the limit a number. An infinite limit describes unbounded behaviour (a vertical asymptote); the limit does not exist as a real number.
- Finding a horizontal asymptote by substituting a large number instead of comparing the leading terms.

Practice. Questions 3 and 4 follow the style of recent free-response parts (IVT from a table; the squeeze theorem with two bounding functions).

1. Find $\displaystyle\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$.
2. Find $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$.
3. A continuous function f has $f(1) = 4$, $f(3) = -2$ and $f(5) = 6$. What is the least number of solutions that $f(x) = 0$ must have on $1 \le x \le 5$? Justify your answer.
4. For all x near 1, $2x + 1 \le k(x) \le x^2 + 2$. Find $\displaystyle\lim_{x\to 1} k(x)$, and say what else you need to know to conclude that k is continuous at $x = 1$.

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5. Let $f(x) = \dfrac{2x^2 - 8}{x^2 - x - 2}$. Find every vertical asymptote, horizontal asymptote and removable discontinuity.
6. Find k so that $f(x) = x^2 + k$ for $x < 2$ and $f(x) = 3x - 1$ for $x \geq 2$ is continuous at $x = 2$.

Answers.

1. Factor: $\dfrac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$, so the limit is 6.
2. $\dfrac{\sin 3x}{x} = 3 \cdot \dfrac{\sin 3x}{3x} \to 3 \cdot 1 = 3$.
3. At least two. f is continuous on $[1, 3]$ and $[3, 5]$; $f(1) = 4 > 0 > -2 = f(3)$, and $f(3) = -2 < 0 < 6 = f(5)$, so by the IVT there is a zero in $(1, 3)$ and another in $(3, 5)$.
4. Both bounds equal 3 at $x = 1$ (and $x^2 + 2 - (2x + 1) = (x - 1)^2 \geq 0$), so by the squeeze theorem $\lim_{x \to 1} k(x) = 3$. k is continuous at 1 only if also $k(1) = 3$.
5. $f(x) = \dfrac{2(x-2)(x+2)}{(x-2)(x+1)}$. Removable discontinuity (hole) at $x = 2$, where the limit is $\frac{2 \cdot 4}{3} = \frac{8}{3}$; vertical asymptote $x = -1$; horizontal asymptote $y = 2$ (ratio of leading coefficients).
6. Left limit $4 + k$ must equal $f(2) = 5$, so $k = 1$.

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Differentiation: Definition and Fundamental Properties

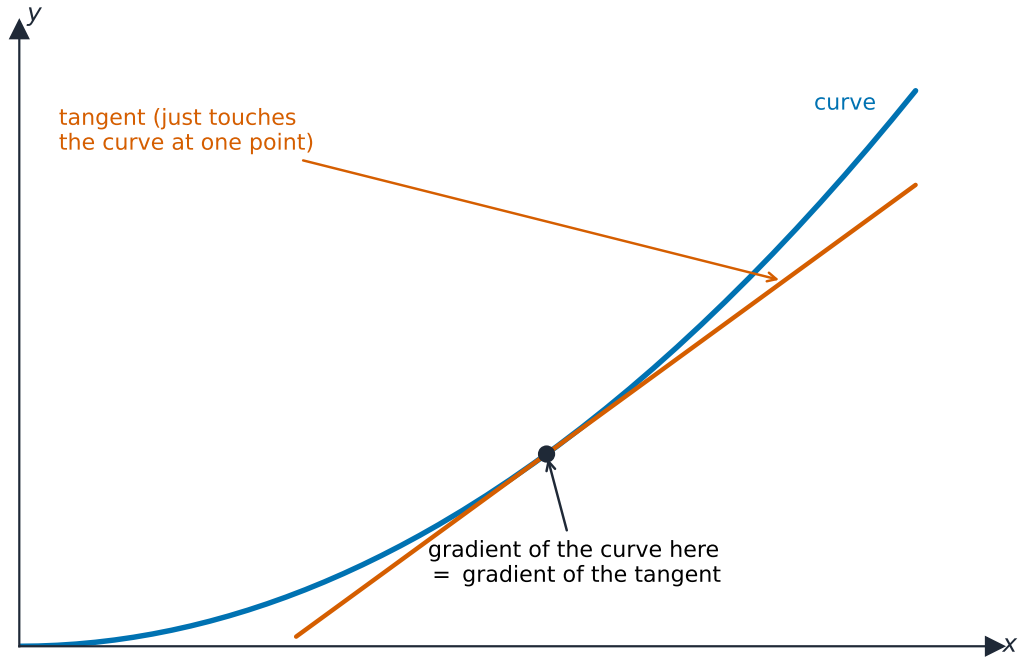
AP Calculus AB

Average and Instantaneous Rates of Change



A roller coaster on a lift hill: the derivative measures instantaneous rate of change of position

Unit 1 built the limit. Unit 2 uses it to define the **derivative** 导数 – the exact rate of change at a point.



The instantaneous rate of change is the gradient of the tangent at a point

Over an interval, the average rate of change is a **difference quotient** 差商. Two equivalent forms appear:

$$\frac{f(a+h)-f(a)}{h} \qquad \text{and} \qquad \frac{f(x)-f(a)}{x-a}.$$

The first uses a step of size h from a; the second uses two points x and a. Both compute $\dfrac{\text{change in output}}{\text{change in input}}$ over the interval.

The **instantaneous** 瞬时 rate of change at $x = a$ is what the difference quotient approaches as the interval shrinks to zero. This limit *is* the derivative at a, written $f'(a)$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

provided the limit exists.

Defining the Derivative and Its Notation

Let the point a vary and the derivative becomes a **new function**:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

This is the **definition of the derivative** (sometimes called differentiating "by first principles" 用定义求导). Its value at each x is the instantaneous rate of change there.

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Worked example. Use the definition to find $f'(x)$ for $f(x) = x^2 + 3x$.

Write $f(x+h)$ in full, subtract $f(x)$, and simplify *before* letting $h \to 0$:

$$f(x+h) - f(x) = \big(x^2 + 2xh + h^2 + 3x + 3h\big) - \big(x^2 + 3x\big) = 2xh + h^2 + 3h.$$

$$f'(x) = \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h\to 0} (2x + h + 3) = 2x + 3.$$

Every term without h cancels, so you can divide by h; that is what removes the $\frac{0}{0}$.
Common **notations** 记号 for the derivative of $y = f(x)$ are:

$$\frac{dy}{dx}, \qquad f'(x), \qquad y'.$$

The derivative can be represented graphically, numerically, analytically, and verbally – be ready to move between them.

Geometric meaning. The derivative at a point is the **slope** 斜率 of the **tangent line** 切线 to the graph there. So the tangent line at $x = a$ passes through $\big(a, f(a)\big)$ with slope $f'(a)$:

$$y - f(a) = f'(a)\,(x - a).$$

Writing this line is a routine exam task, so keep the point-slope form ready.

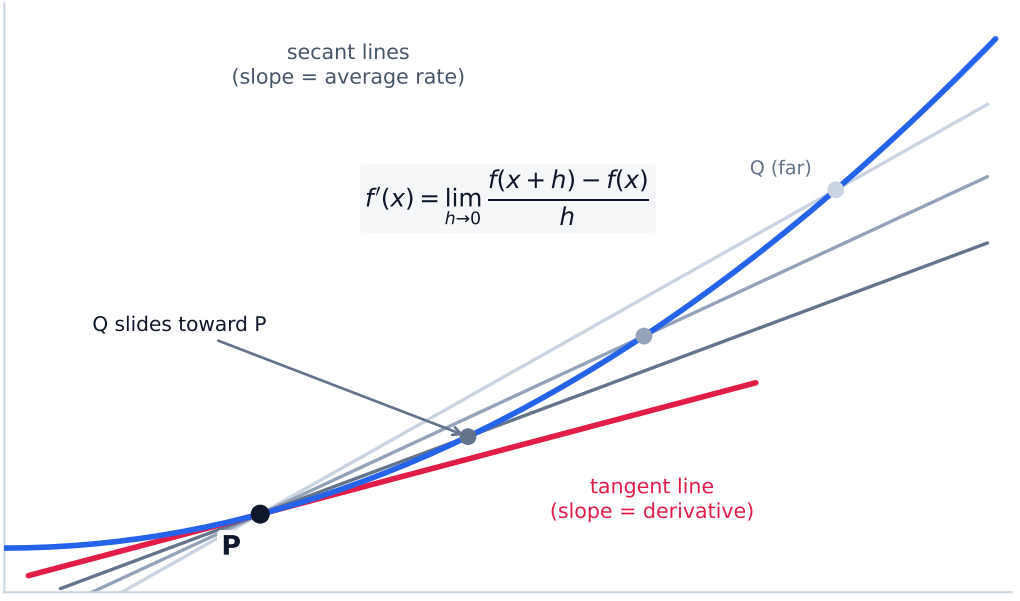
Worked example. Find the tangent line to $f(x) = x^3 - 2x$ at $x = 2$.

- The point: $f(2) = 2^3 - 2(2) = 8 - 4 = 4$, so the line passes through $(2, 4)$.
- The slope: $f'(x) = 3x^2 - 2$, so $f'(2) = 3(4) - 2 = 10$.
- The line: $y - 4 = 10(x - 2)$, or $y = 10x - 16$.

Use the derivative only for the slope; the point comes from f itself.

Common mistakes. Using $f'(2) = 10$ as the y-coordinate of the point, and leaving x in the slope: $y - 4 = (3x^2 - 2)(x - 2)$ is not a line. The slope of a tangent line is a **number**, $f'(a)$.

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As Q slides toward P, each secant's slope (an average rate) approaches the tangent's slope – the derivative $f'(a)$.

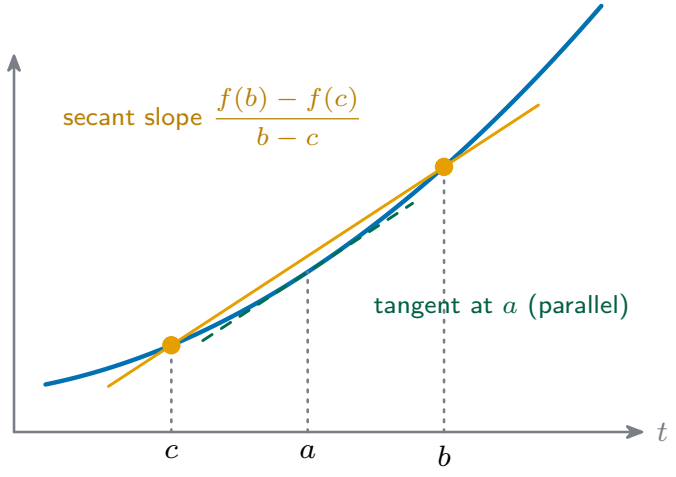
Estimating a Derivative at a Point

You do not always have a formula. When a function is given by a **table** 表格 or a graph, estimate the derivative $f'(a)$ with a difference quotient over a **small interval around** a. A table with values on both sides of a gives the best estimate:

$$f'(a) \approx \frac{f(b) - f(c)}{b - c}, \qquad \text{where } c < a < b \text{ are the closest table inputs.}$$

Technology (a calculator) can also estimate a derivative at a point.

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Bracket the point with the closest table inputs.

Exam skill (appears almost every year). Questions such as "Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$" ask for exactly this difference quotient. Show the setup:

$$M'(7.5) \approx \frac{M(10) - M(5)}{10 - 5}.$$

Full credit needs the numbers plugged in **and** the correct **units** 单位 (output units per input unit), since these come from real-world models.

> **Worked example.** Water is pumped into a tank. $M(t)$ is the volume in the tank, in litres, t minutes after the pump starts.

t **(minutes)**	0	5	10	15
$M(t)$ (litres)	12	20	26	29

Estimate $M'(7.5)$. The closest table inputs on each side of 7.5 are 5 and 10:

$$M'(7.5) \approx \frac{M(10) - M(5)}{10 - 5} = \frac{26 - 20}{5} = 1.2 \text{ litres per minute.}$$

Meaning: about 7.5 minutes after the start, the volume is rising at about 1.2 litres per minute.

Common mistakes. Writing $M(10) - M(5) = 6$ and forgetting to divide by $10 - 5$; dividing by the wrong difference; leaving out the units (litres **per minute**); and using a one-sided interval such as $5 \le t \le 15$ when the table has the closest values on both sides of 7.5.

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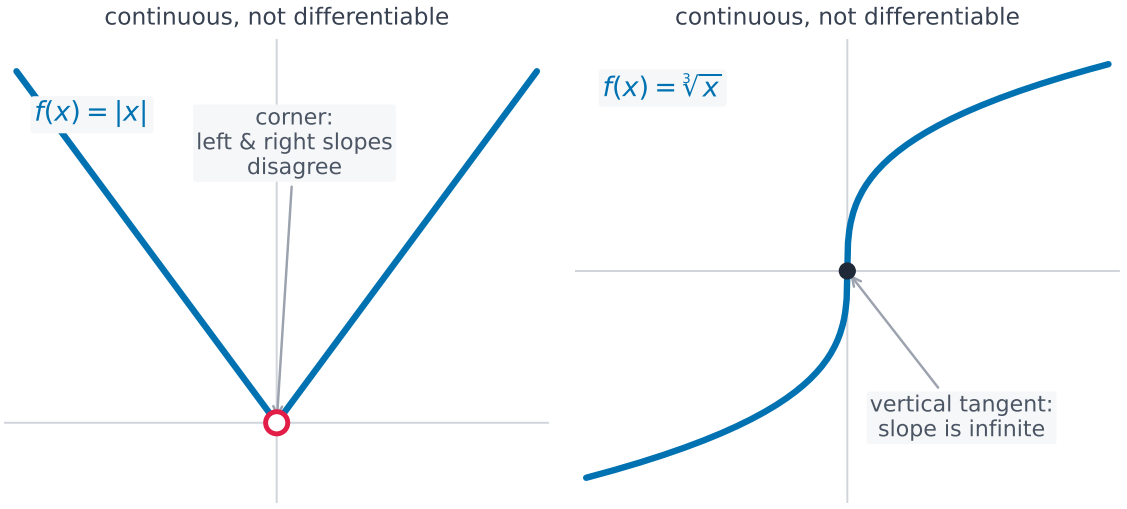
When Does a Derivative Exist?

Differentiability is stronger than continuity. The key relationship:

> **If f is differentiable 可导 at a point, then f is continuous 连续 there.**

So differentiability *implies* continuity. The reverse is false: a continuous function can fail to be differentiable. Two ways this happens:

- A **corner** 尖角: the left and right difference-quotient limits disagree, as with $f(x) = |x|$ at $x = 0$.
- A **vertical tangent** 垂直切线: the slope is infinite (no real number), as with $f(x) = \sqrt[3]{x}$ at $x = 0$.



Two ways a continuous function is not differentiable: a corner and a vertical tangent

Also, a point outside the domain of f cannot be in the domain of f'. Use the contrapositive on the exam: **if f is not continuous at a, then f is not differentiable at a.**

Common mistake. Arguing "continuous, so differentiable". The function $|x|$ is continuous at 0 but has a corner there. Continuity is **needed** for differentiability, but it is not enough.

> **Worked example (making a piecewise function differentiable).** Find k and m so that
>
> $$f(x) = \begin{cases} x^2 + k, & x \le 2 \\ mx - 1, & x > 2 \end{cases}$$
>
> is differentiable at $x = 2$. Two conditions are needed.

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1. *Continuous:* the pieces meet, $2^2 + k = 2m - 1$, so $k = 2m - 5$.
2. *Same slope from both sides:* the left piece has slope $2x = 4$ at $x = 2$; the right piece has slope m. So $m = 4$, and then $k = 3$.

Check: both pieces give $f(2) = 7$ and slope 4. Matching the slopes alone is not enough: without condition 1, f would jump at $x = 2$ and could not be differentiable.

The Power Rule

From here we use **rules** instead of the limit definition each time. The **power rule** 幂法则 handles any power of x:

$$\frac{d}{dx}\, x^r = r\, x^{\,r-1} \qquad \text{for any real } r.$$

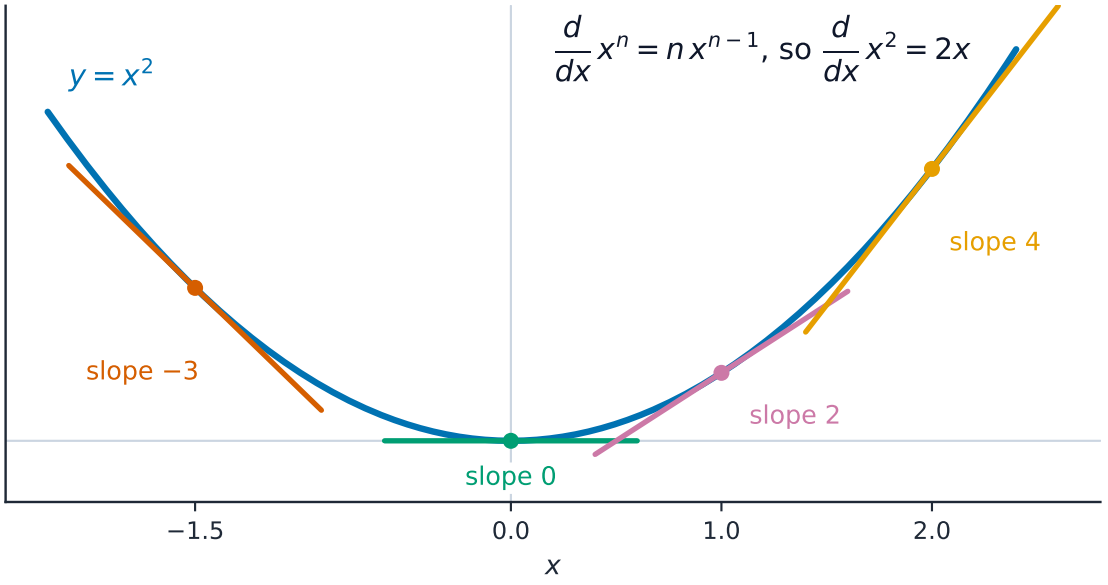
For arbitrary real powers this formula applies on $x > 0$; integer powers and some rational powers have larger real domains. Check the domain and differentiability at the point, especially at $x = 0$. Rewrite roots as powers before applying the rule.

Worked example. Differentiate $y = 6\sqrt{x} - \dfrac{2}{x^3} + \dfrac{1}{3x}$. Rewrite every term as a constant times a power of x first:

$$y = 6x^{1/2} - 2x^{-3} + \tfrac{1}{3}x^{-1} \quad \Rightarrow \quad y' = 3x^{-1/2} + 6x^{-4} - \tfrac{1}{3}x^{-2} = \frac{3}{\sqrt{x}} + \frac{6}{x^4} - \frac{1}{3x^2}.$$

Common mistakes. Writing $\dfrac{d}{dx}\dfrac{1}{x^3} = \dfrac{1}{3x^2}$ (differentiating the denominator), and treating $\dfrac{1}{3x}$ as $3x^{-1}$: the 3 stays in the denominator, so the constant is $\frac{1}{3}$.

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On $y = x^2$ the tangent slope at x is always $2x$

Constant, Sum, Difference, and Constant Multiple Rules

These rules let you differentiate term by term:

- **Constant:** $\dfrac{d}{dx}\,k = 0$ (a constant does not change).
- **Constant multiple** 常数倍: $\dfrac{d}{dx}\big[k\,f(x)\big] = k\,f'(x)$.
- **Sum / difference:** $\dfrac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x)$.

Combined with the power rule, they differentiate any **polynomial** 多项式 term by term. Example:

$$\frac{d}{dx}\big(4x^3 - 5x + 7\big) = 12x^2 - 5.$$

> **Worked example (horizontal tangents).** Where does $y = x^3 - 6x^2 + 9x + 1$ have a horizontal tangent? Set the derivative to zero: $y' = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$, so $x = 1$ or $x = 3$. The points are $(1, 5)$ and $(3, 1)$. Give the **points**, not just the x-values, when the question asks "where on the curve".

Derivatives of sin x, cos x, e^x, and ln x

Learn these four building-block derivatives by heart:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

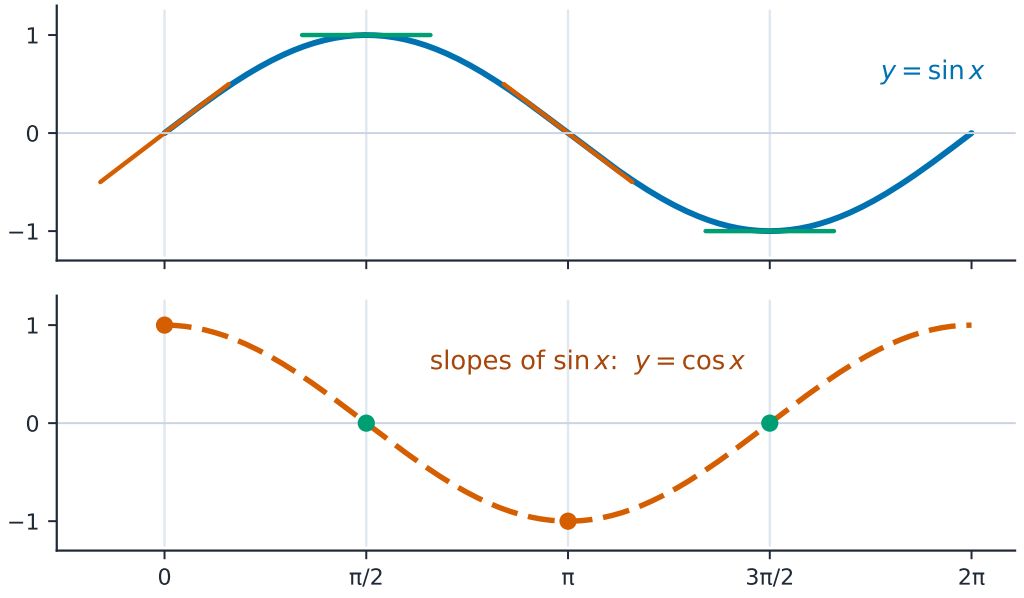
$$\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0).$$

Note the minus sign on the derivative of cosine, and that e^x is its own derivative.

A limit that is really a derivative (LIM-3.A.1). Sometimes a limit is secretly the definition of a known derivative. If you recognize

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

for a function f whose derivative you know, just evaluate $f'(a)$. For example, $\lim_{h\to 0}\dfrac{\sin\left(\frac{\pi}{2}+h\right)-1}{h} = \dfrac{d}{dx}\sin x\Big|_{x=\pi/2} = \cos\frac{\pi}{2} = 0.$



The slopes of sin x trace out cos x

The Product Rule

A product of two functions is **not** differentiated by multiplying the derivatives. Use the **product rule** 乘积法则:

$$\frac{d}{dx}\big[u\,v\big] = u'v + uv'.$$

"Derivative of the first times the second, plus the first times the derivative of the second." Example:

$$\frac{d}{dx}\left(x^2e^x\right) = 2x\,e^x + x^2e^x.$$

Exam questions often build a new function from given pieces, e.g. $k(x) = \big(f(x)\big)^2 g(x)$, and ask for $k'(x)$ at a point, you to combine rules while reading values from a table.

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The Quotient Rule

For a quotient, use the **quotient rule** 商法则:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}.$$

"Bottom times derivative of top, minus top times derivative of bottom, all over bottom squared." The order matters because of the minus sign, so write the numerator carefully. Example:

$$\frac{d}{dx}\left(\frac{x}{\cos x}\right) = \frac{1\cdot\cos x - x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos x + x\sin x}{\cos^2 x}.$$

Worked example: rules from a table. You are given $f(3) = 2$, $f'(3) = -1$, $g(3) = 5$ and $g'(3) = 4$.

- For $p(x) = f(x)\,g(x)$: $p'(3) = f'(3)g(3) + f(3)g'(3) = (-1)(5) + (2)(4) = 3$.
- For $q(x) = \dfrac{f(x)}{g(x)}$:

$$q'(3) = \frac{f'(3)g(3) - f(3)g'(3)}{\big(g(3)\big)^2} = \frac{(-1)(5) - (2)(4)}{5^2} = -\frac{13}{25}.$$

Common mistakes. Multiplying the derivatives: $p'(3) \neq f'(3)\,g'(3) = -4$. Writing the quotient-rule numerator in the wrong order ($fg' - f'g$ gives $+\frac{13}{25}$, the wrong sign). Forgetting to square the denominator.

Derivatives of tan x, cot x, sec x, and csc x

The remaining trigonometric derivatives are not memorized separately – you rewrite them with **identities** 恒等式 and apply the quotient (or product) rule. For instance, $\tan x = \dfrac{\sin x}{\cos x}$, so the quotient rule gives

$$\frac{d}{dx}\tan x = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The same method (writing $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$) gives $-\csc^2 x$, $\sec x\tan x$, and $-\csc x\cot x$.

Worked example. Find the tangent line to $y = \tan x$ at $x = \frac{\pi}{4}$. The point is $\left(\frac{\pi}{4}, 1\right)$ and the slope is $\sec^2\frac{\pi}{4} = \left(\sqrt{2}\right)^2 = 2$, so $y - 1 = 2\left(x - \frac{\pi}{4}\right)$.

A memory aid: the three "co" functions (cot, csc, and cos) all have a **minus sign** in their derivatives.

Higher-order derivatives. Differentiating f' again gives the **second derivative** 二阶导数 $f''(x)$ (or $\frac{d^2y}{dx^2}$) – the rate of change of the rate of change. An exam part like

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"Find $k''(3)$" just means differentiate twice, then substitute. You can also estimate a second derivative from a table by applying the average-rate-of-change method to the f' values.

> **Worked example.** Differentiate $f(x) = x^2 \sin x$ with the **product rule** $(uv)' = u'v + uv'$: take $u = x^2$ $(u' = 2x)$ and $v = \sin x$ $(v' = \cos x)$, giving $f'(x) = 2x \sin x + x^2 \cos x$. Always read the *structure* first – a product needs the product rule, not the power rule applied to each factor separately.

Exam tips

- The derivative is the **slope of the tangent** – the limit of the secant slope $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$.
- Memorise the rules: power, product, quotient, and the derivatives of sin, cos, e^x, and $\ln x$.
- **Differentiability implies continuity**, but not the reverse (a corner or cusp is continuous yet not differentiable).
- Distinguish average rate of change (secant slope over an interval) from instantaneous rate (the derivative at a point).
- Give a tangent-line equation as $y - f(a) = f'(a)(x - a)$.

Practice.

1. Use the definition of the derivative to find $f'(x)$ for $f(x) = 3x^2 - x$.
2. A table gives $h(4) = 10$ and $h(7) = 19$. Estimate $h'(5.5)$ and state its units if h is in metres and t in seconds.
3. Differentiate $y = \dfrac{4}{\sqrt{x}} + x^5 - \pi^2$.
4. Find k and m so that $g(x) = kx^2$ for $x \le 1$ and $g(x) = 4x + m$ for $x > 1$ is differentiable at $x = 1$.
5. Differentiate $y = e^x \cos x$ and $y = \dfrac{\ln x}{x}$.
6. Find the equation of the tangent line to $y = \sec x$ at $x = 0$.

Answers.

1. $\dfrac{3(x+h)^2 - (x+h) - 3x^2 + x}{h} = \dfrac{6xh + 3h^2 - h}{h} = 6x + 3h - 1 \to 6x - 1.$
2. $\dfrac{19 - 10}{7 - 4} = 3$ metres per second.
3. $y = 4x^{-1/2} + x^5 - \pi^2$, so $y' = -2x^{-3/2} + 5x^4$ (π^2 is a constant).
4. Slopes: $2k = 4$, so $k = 2$. Continuity: $k = 4 + m$, so $m = -2$.
5. $e^x \cos x - e^x \sin x$; $\dfrac{\frac{1}{x} \cdot x - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$.
6. $y' = \sec x \tan x = 0$ at $x = 0$, and $y(0) = 1$, so $y = 1$.

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Differentiation: Composite, Implicit, and Inverse Functions

AP Calculus AB

The Chain Rule

Unit 2 differentiated single functions. Unit 3 differentiates functions built *inside* other functions. The **chain rule** 链式法则 differentiates a **composite function** 复合函数 $f\big(g(x)\big)$:

$$\frac{d}{dx}\,f\big(g(x)\big) = f'\big(g(x)\big)\cdot g'(x).$$

"Derivative of the outer function (leaving the inside alone), times the derivative of the inside." The inner derivative $g'(x)$ is the piece students forget, so always ask "what is the inside, and what is its derivative?" Example:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot 2x.$$

In **Leibniz notation** 莱布尼茨记号, with $y = f(u)$ and $u = g(x)$, the rule reads $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$ – the intermediate du appears to "cancel." Exam questions often give a table for f, g, f', g' and ask for $h'(a)$ where $h(x) = f\big(g(x)\big)$; evaluate $f'\big(g(a)\big)\cdot g'(a)$ by reading values.

> **Worked example.** Differentiate $h(x) = (2x^2+1)^5$. The **outer function** 外函数 is "(something)^5", the **inner function** 内函数 is $2x^2+1$:
>
> $$h'(x) = 5(2x^2+1)^4\cdot\frac{d}{dx}(2x^2+1) = 5(2x^2+1)^4\cdot 4x = 20x(2x^2+1)^4.$$

> **Worked example (from a table).** The table gives values of f, f', g and g'.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	4	-2	3	5
2	1	6	-1	2
3	7	3	2	-4

For $k(x) = f\big(g(x)\big)$, find the tangent line at $x = 3$. The point is $k(3) = f\big(g(3)\big) = f(2) = 1$. The slope is $k'(3) = f'\big(g(3)\big)\,g'(3) = f'(2)(-4) = 6(-4) = -24$. So the tangent is $y - 1 = -24(x-3)$.

For $p(x) = f(x^2 - x)$ at $x = 2$: the inside is $x^2 - x = 2$ and its derivative is $2x - 1 = 3$, so $p'(2) = f'(2)\cdot 3 = 18$. For $h(x) = \ln\big(f(x)\big)$: $h'(x) = \dfrac{f'(x)}{f(x)}$, so $h'(3) = \dfrac{3}{7}$. For a product with a chain inside, $m(x) = f(2x)\,g(x)$, use both rules: $m'(x) = 2f'(2x)\,g(x) +$

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$f(2x)\,g'(x)$, so $m'(1) = 2f'(2)\,g(1) + f(2)\,g'(1) = 2(6)(3) + 1(5) = 41$. The inner derivative 2 from $f(2x)$ is the factor most often lost.



Matryoshka dolls: the chain rule differentiates nested functions layer by layer – outer first, then the next inside

Implicit Differentiation

Some curves are defined **implicitly** – by an equation in x and y that is not solved for y, such as $x^2 + y^2 = 25$. **Implicit differentiation** 隐函数求导 finds $\dfrac{dy}{dx}$ without solving for y first. It is just the chain rule, treating y as a function of x.

The method: differentiate **both sides** with respect to x; every time you differentiate a y-term, multiply by $\dfrac{dy}{dx}$ (the chain rule); then solve algebraically for $\dfrac{dy}{dx}$. For $x^2+y^2 = 25$:

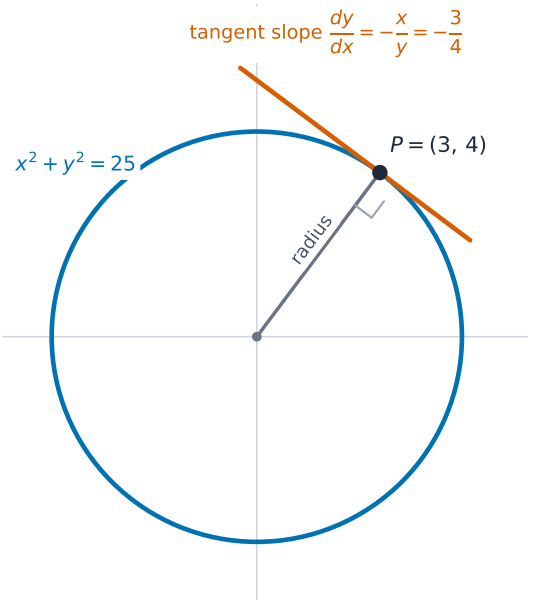
$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

Worked example. Find the **tangent line** 切线 to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. From above $\dfrac{dy}{dx} = -\dfrac{x}{y} = -\dfrac{3}{4}$ there, so the tangent is

$$y - 4 = -\tfrac{3}{4}(x - 3).$$

Notice the tangent is **perpendicular** 垂直 to the **radius** 半径, as geometry promises – a good sanity check.

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Implicit differentiation gives the tangent to a circle, perpendicular to the radius

Exam skill – "Show that $\frac{dy}{dx} = \ldots$". This exact prompt appears most years (e.g. "Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$"). Because the target is given, you must show **every algebra step** cleanly: differentiate both sides, use the product/chain rules on mixed xy terms, collect all $\frac{dy}{dx}$ terms on one side, factor, and divide. A correct final line that skips the algebra earns little. Follow-up parts then ask for a tangent line, or where the tangent is horizontal (the numerator is 0 and the denominator is nonzero) or vertical (the denominator is 0 and the numerator is nonzero). If both are zero, these tests are inconclusive.

> **Worked example.** The curve $x^2 + xy + y^2 = 7$ passes through $(1, 2)$. Show that $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$, and find the slope at $(1, 2)$.

The middle term xy is a product, so it needs the product rule: $\frac{d}{dx}(xy) = 1 \cdot y + x\frac{dy}{dx}$. Differentiate every term:

$$2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0.$$

Collect the $\frac{dy}{dx}$ terms on one side and factor:

$$(x + 2y)\frac{dy}{dx} = -(2x + y) \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

At $(1, 2)$ (check: $1 + 2 + 4 = 7$), the slope is $-\dfrac{2(1)+2}{1+2(2)} = -\dfrac{4}{5}$.

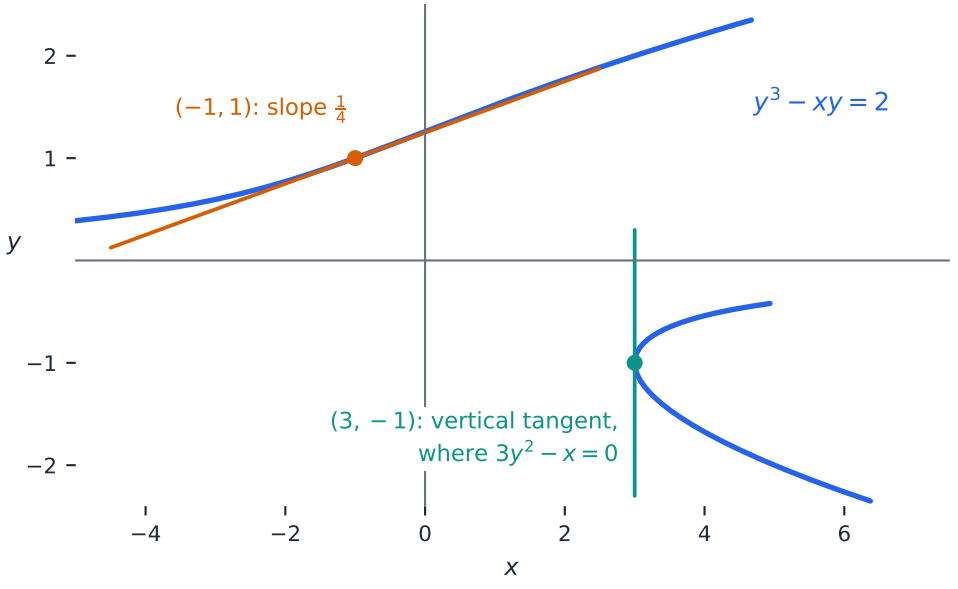
Common mistakes. Writing $\dfrac{d}{dx}(xy) = \dfrac{dy}{dx}$ (the product rule is missing), and writing $\dfrac{d}{dx}(y^2) = 2y$ (the factor $\dfrac{dy}{dx}$ is missing).

Worked example (horizontal and vertical tangents). For the curve $y^3 - xy = 2$:

$$3y^2\frac{dy}{dx} - \left(y + x\frac{dy}{dx}\right) = 0 \implies \frac{dy}{dx} = \frac{y}{3y^2 - x}.$$

At $(-1, 1)$ the slope is $\dfrac{1}{3+1} = \dfrac{1}{4}$, so the tangent is $y - 1 = \frac{1}{4}(x + 1)$.

- **Horizontal** tangent: numerator $y = 0$. But $y = 0$ gives $0 - 0 = 2$, which is false, so no point on the curve has a horizontal tangent.
- **Vertical** tangent: denominator $3y^2 - x = 0$, so $x = 3y^2$. Substitute into the curve: $y^3 - 3y^3 = 2$, so $y^3 = -1$, $y = -1$ and $x = 3$. The tangent is vertical at $(3, -1)$; check the numerator there is not also 0 ($y = -1 \neq 0$).



Where the denominator of $\frac{dy}{dx}$ is zero (and the numerator is not), the tangent is vertical

Always substitute back into the **original** equation: a point that makes the derivative zero or undefined is useless unless it is on the curve.

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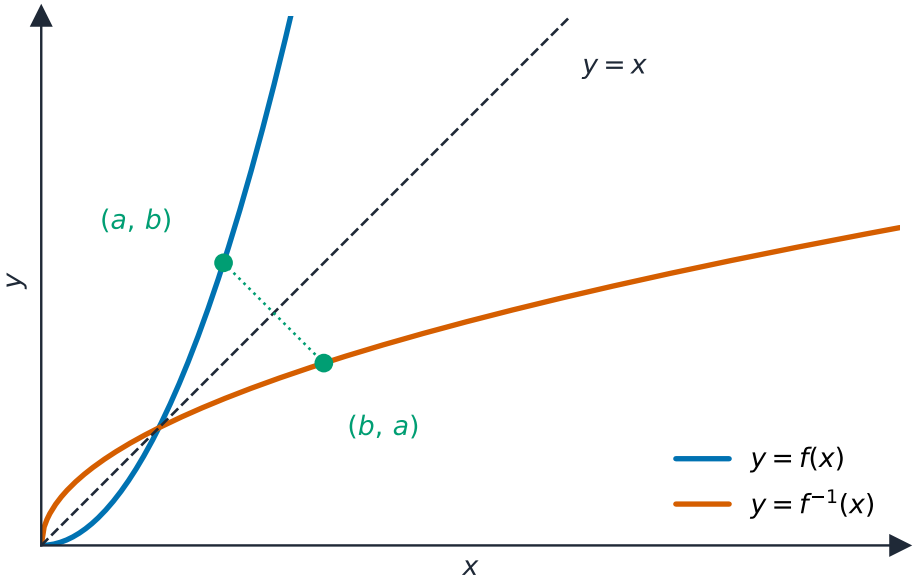
Differentiating Inverse Functions

If g is the **inverse function** 反函数 of f (so $f(g(x)) = x$), the chain rule links their derivatives:

$$g'(x) = \frac{1}{f'\big(g(x)\big)}, \qquad \text{provided } f'\big(g(x)\big) \neq 0.$$

In words: the derivative of the inverse at a point is the **reciprocal** 倒数 of the derivative of the original function at the *matching* point. A common exam setup gives a table and a point (a, b) on f (so (b, a) is on g), then asks for $g'(b) = \dfrac{1}{f'(a)}$.

You do not need a formula for the inverse. For $f(x) = x^3 + x$, let $g = f^{-1}$; to find $g'(2)$, first solve $f(a) = 2$: $a^3 + a = 2$ gives $a = 1$. Then $g'(2) = \dfrac{1}{f'(1)} = \dfrac{1}{3(1)^2 + 1} = \dfrac{1}{4}$.



The inverse function is the mirror image of the function in the line $y = x$

Worked example. Suppose $f(2) = 5$ and $f'(2) = 3$, and g is the inverse of f. Because $(2, 5)$ lies on f, the point $(5, 2)$ lies on g, and

$$g'(5) = \frac{1}{f'(2)} = \frac{1}{3}.$$

The graphs of f and g are mirror images across $y = x$, so their slopes at matching points are reciprocals.

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Differentiating Inverse Trigonometric Functions

The same idea gives the derivatives of the **inverse trigonometric functions** 反三角函数. The three you should know:

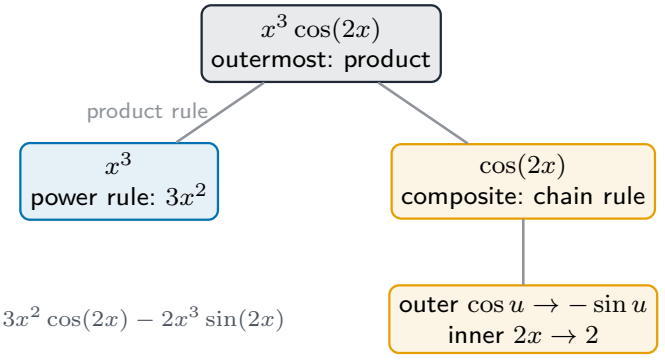
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

Combine these with the chain rule when the input is itself a function, e.g. $\frac{d}{dx}\arctan(3x) = \frac{3}{1+9x^2}$. The co-functions are the negatives: $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$, and similarly for arccot and arccsc.

> **Worked example.** $\frac{d}{dx}\arcsin(x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$. Where these formulas come from: if $y = \arcsin x$, then $\sin y = x$; differentiating implicitly, $\cos y \frac{dy}{dx} = 1$, so $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$ (using $\cos y = \sqrt{1-\sin^2 y}$, positive for $-\frac{\pi}{2} < y < \frac{\pi}{2}$).

Selecting Procedures for Calculating Derivatives

A skill topic: real derivatives mix several rules, so read the structure of the expression first, from the outside in.



Name the outermost operation, then recurse inward.

- Is it a **sum**? Differentiate term by term.
- A **product** or **quotient**? Apply the **product rule** 乘积法则 or the **quotient rule** 商法则, and expect to use the chain rule inside.
- A **composite** (something inside something)? Chain rule.
- Given **implicitly**? Implicit differentiation.

Name the outermost operation, apply its rule, and recurse inward. Neatness prevents the sign and bookkeeping errors that cost marks.

Worked example (choosing the rules). Differentiate (a) $y = x^3\cos(2x)$ and (b) $y = \ln\left(x^2 e^x\right)$ for $x > 0$.

(a) The outermost operation is a **product** of x^3 and $\cos(2x)$, and the second factor is a composite. Product rule, with the chain rule inside:

$$\frac{dy}{dx} = 3x^2\cos(2x) + x^3\cdot\big(-\sin(2x)\big)\cdot 2 = 3x^2\cos(2x) - 2x^3\sin(2x).$$

(b) The outermost operation is ln of a product. The chain rule works, but the **laws of logarithms** 对数运算法则 make it shorter: $\ln\left(x^2e^x\right) = 2\ln x + x$, so

$$\frac{dy}{dx} = \frac{2}{x} + 1.$$

Simplifying first is allowed whenever the new form is equal for every x in the domain. It also removes chances to slip: the chain-rule route gives $\dfrac{2xe^x + x^2e^x}{x^2e^x}$, which is the same answer only after cancelling.

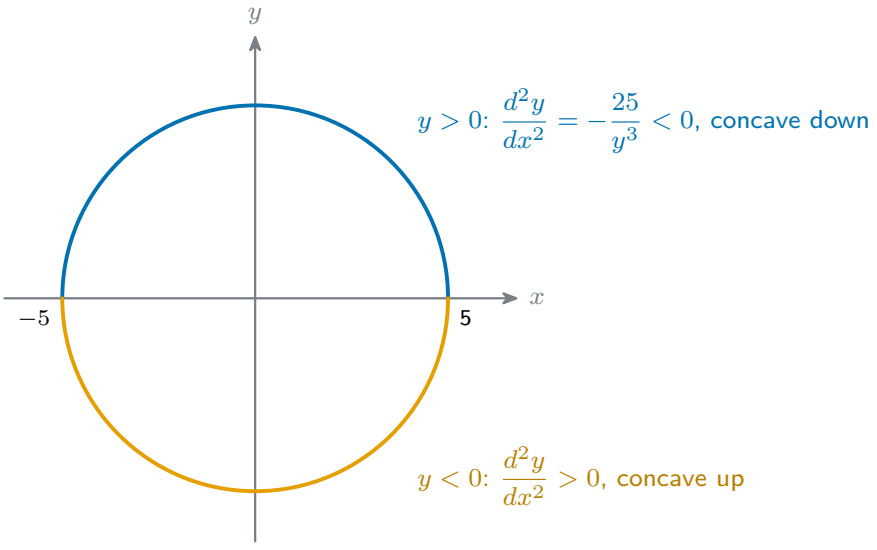
Calculating Higher-Order Derivatives

Differentiating f' produces the **second derivative** 二阶导数 f''; repeating gives **higher-order derivatives** 高阶导数. The notations:

$$f''(x) = \frac{d^2y}{dx^2} = y'', \qquad \text{and in general} \qquad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

The second derivative measures how the *slope* is changing; it drives **concavity** 凹凸性 and **acceleration** 加速度 in later units. To find f'' implicitly, differentiate the expression for $\dfrac{dy}{dx}$ again (with the quotient and chain rules), then substitute $\dfrac{dy}{dx}$ back in.

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The sign of the second derivative matches the circle's shape.

Worked example. For the circle $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$ in terms of y.

Start from $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and use the quotient rule, remembering that y depends on x:

$$\frac{d^2y}{dx^2} = -\frac{1 \cdot y - x\dfrac{dy}{dx}}{y^2} = -\frac{y - x\left(-\dfrac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}.$$

The last step uses the curve itself: $x^2 + y^2 = 25$. On the top half ($y > 0$) the second derivative is negative, so the curve is concave down there, as a circle's top should be.

Worked example (second derivative at a point). For $y^3 - xy = 2$ at $(-1, 1)$ we found $\dfrac{dy}{dx} = \dfrac{1}{4}$. Differentiate $\dfrac{dy}{dx} = \dfrac{y}{3y^2 - x}$ with the quotient rule:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{dy}{dx}(3y^2 - x) - y\left(6y\dfrac{dy}{dx} - 1\right)}{(3y^2 - x)^2}.$$

Substitute numbers **before** simplifying: $\dfrac{\frac{1}{4}(4) - 1\left(6 \cdot \frac{1}{4} - 1\right)}{4^2} = \dfrac{1 - \frac{1}{2}}{16} = \dfrac{1}{32}$.

From a differential equation. If $\dfrac{dy}{dx} = 2x - y$, then $\dfrac{d^2y}{dx^2} = 2 - \dfrac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$. In Quadrant II ($x < 0$, $y > 0$), $2 - 2x + y > 0$, so every solution curve is concave up there.

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Practice.

1. Differentiate $y = \cos^3(2x)$.
2. Use the table above: find $\dfrac{d}{dx}\, g\big(f(x)\big)$ at $x = 2$.
3. For $x^2y + y^3 = 10$, find $\dfrac{dy}{dx}$ at $(3, 1)$.
4. $f(x) = e^x + 2x$ and $g = f^{-1}$. Find $g'(1)$.
5. Find $\dfrac{d}{dx} \arctan\left(\sqrt{x}\right)$.
6. Find $\dfrac{d^2y}{dx^2}$ for $y = x \ln x$.

Answers.

1. $3\cos^2(2x) \cdot (-\sin 2x) \cdot 2 = -6\cos^2(2x)\sin(2x)$.
2. $g'\big(f(2)\big)\, f'(2) = g'(1) \cdot 6 = 5 \cdot 6 = 30$.
3. $2xy + x^2\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2 + 3y^2} = -\dfrac{6}{12} = -\dfrac{1}{2}$.
4. $f(0) = 1$, so $g(1) = 0$ and $g'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{e^0 + 2} = \dfrac{1}{3}$.
5. $\dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}\,(1+x)}$.
6. $y' = \ln x + 1$, so $y'' = \dfrac{1}{x}$.

Exam tips

- Use the **chain rule** for composite functions – differentiate the outside, then multiply by the derivative of the inside (the most-forgotten factor).
- For **implicit** differentiation, differentiate both sides with respect to x and attach $\frac{dy}{dx}$ each time y is differentiated, then solve.
- Get the **second derivative** by differentiating twice (velocity $\to$ acceleration).
- Combine rules carefully in layered expressions (chain inside product, etc.).
- The inverse function's graph is the reflection in $y = x$; its slope is the reciprocal of the original's at the matching point.
- Horizontal tangent: numerator of $\frac{dy}{dx}$ is 0 and denominator is nonzero; vertical tangent: denominator is 0 and numerator is nonzero. If both vanish, investigate further. Check the point is on the curve.
- For an implicit second derivative at a point, substitute the numbers (including $\frac{dy}{dx}$) before simplifying.

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Contextual Applications of Differentiation

AP Calculus AB

Interpreting the Derivative in Context

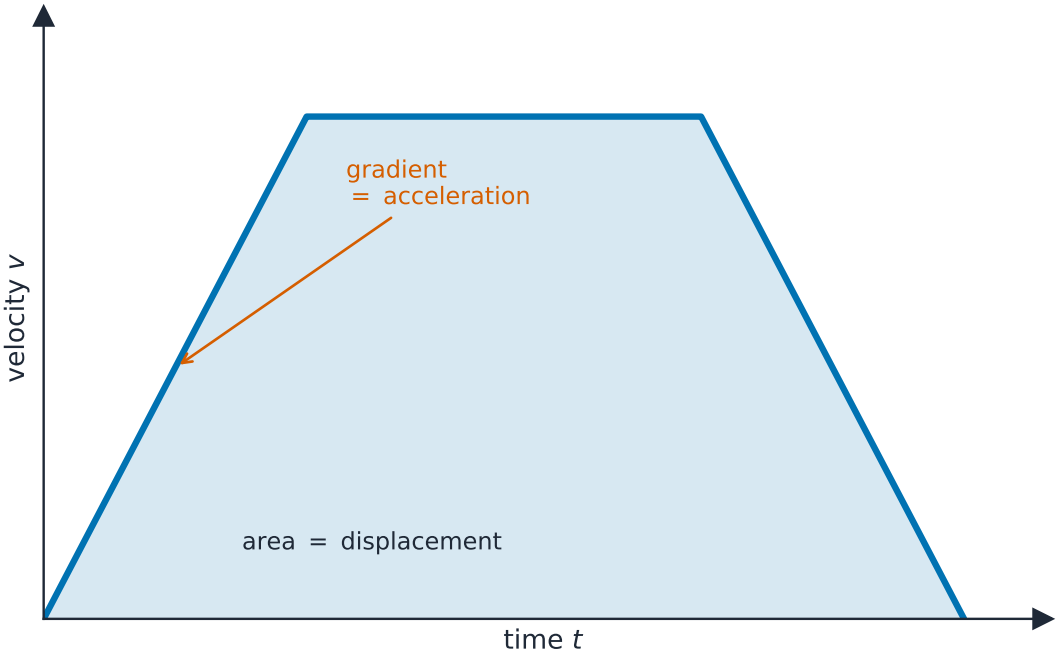
Once you can compute derivatives, you use them to describe the real world. The derivative $f'(x)$ **is** the **instantaneous rate of change** 瞬时变化率 of f **with respect to** 关于 its input. Reading and reporting this rate correctly is a graded skill.

Units matter. The unit of $f'(x)$ is the unit of f divided by the unit of x. If $C(t)$ is a number of acres and t is in weeks, then $C'(t)$ is in acres per week. On the exam, "Using correct units, interpret the meaning of $g'(140)$" wants a full sentence: the *value*, the *quantity*, the *rate word* "per", and the *moment*. For example: "$g'(140) = 2.3$ means that at $x = 140$, the quantity is increasing at about 2.3 units per unit of x."

Common mistakes. Reading $g'(140)$ as an amount ("the quantity is 2.3") instead of a rate; leaving out the units or the moment ($x = 140$); and calling a negative derivative "increasing".

Straight-Line Motion: Position, Velocity, and Acceleration

For a **particle** 质点 moving on a line, three functions of time are linked by **differentiation** 求导:



On a velocity-time graph, the area is displacement and the gradient is acceleration

- **position** 位置 $s(t)$;

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- **velocity** 速度 $v(t) = s'(t)$ – signed; its sign gives direction;
- **acceleration** 加速度 $a(t) = v'(t) = s''(t)$.

Key readings (frequent exam parts):

- The particle is **at rest** 静止 when $v(t) = 0$.
- It moves **right/up** when $v(t) > 0$ and **left/down** when $v(t) < 0$; it **changes direction** 改变方向 where v changes sign.
- **Speed** 速率 is $|v(t)|$. Speed is *increasing* when v and a have the **same sign** (the particle is **speeding up** 加速), and *decreasing* when they have **opposite signs**.

> **Worked example.** A particle moves with $s(t) = t^3 - 6t^2 + 9t$. Then $v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$, so it is at rest at $t = 1$ and $t = 3$, and changes direction at each. At $t = 2$, $v(2) = 3(1)(-1) = -3 < 0$ (moving left) and $a(2) = 6(2) - 12 = 0$; just after, $a > 0$ while $v < 0$, so it is **slowing down** there. Distinguish carefully between velocity (has direction) and speed (does not) – the exam tests this exact difference.

Common mistakes. Saying the particle speeds up because $a > 0$: compare the signs of **both** v and a. Saying it changes direction wherever $v = 0$: v must change **sign**. With $v(t) = (t-2)^2$ the particle stops for an instant at $t = 2$, then keeps moving the same way.

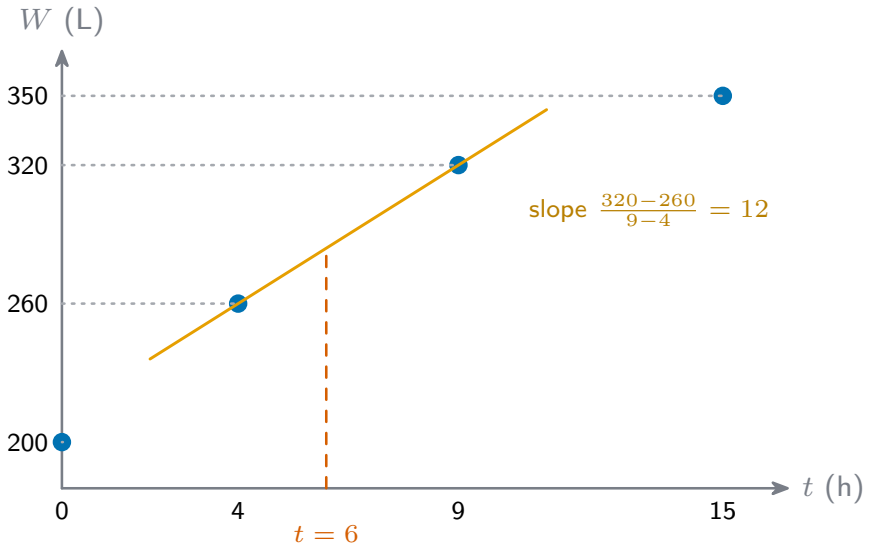


A roller coaster is calculus you can feel: velocity is the derivative of position, and acceleration is the derivative of velocity

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Rates of Change Beyond Motion

The same derivative idea models any changing quantity: a draining tank, a spreading population, a cooling cup. Whenever a problem says "the rate at which...", it is describing a derivative. Read the units to know which quantity's rate you have, then interpret in context.



Use the two measurements that bracket $t = 6$.

Worked example (estimating a rate from a table). Water is pumped into a tank. The volume $W(t)$, in litres, is measured at times t in hours.

t **(hours)**	0	4	9	15
$W(t)$ (litres)	200	260	320	350

Estimate $W'(6)$. Use the two measurements that bracket $t = 6$:

$$W'(6) \approx \frac{W(9) - W(4)}{9 - 4} = \frac{320 - 260}{5} = 12 \text{ litres per hour.}$$

Interpretation: at $t = 6$ hours, the volume of water in the tank is increasing at about 12 litres per hour. A full answer names the quantity, the time, increasing or decreasing, the size and the units.

A rate of a rate. "Is the rate at which water enters increasing at $t = 6$?" asks about the sign of $W''(6)$, not $W'(6)$. Here the rate is 15 L/h on $[0, 4]$, 12 L/h on $[4, 9]$ and 5 L/h on $[9, 15]$, so the data suggest the rate is **decreasing**.

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Introduction to Related Rates

In a **related rates** 相关变化率 problem, several quantities change together over time, and you know some rates but want another. The engine is the **chain rule** 链式法则: differentiate a relationship **with respect to time** t. Every variable becomes a function of t, so each derivative picks up a "$\,/\,dt$" factor. The **product rule** 乘积法则 and **quotient rule** 商法则 may also be needed.



A rising balloon: related rates link how fast radius, volume and height change together

Solving Related Rates Problems

A reliable procedure – and a full-credit template on the exam:

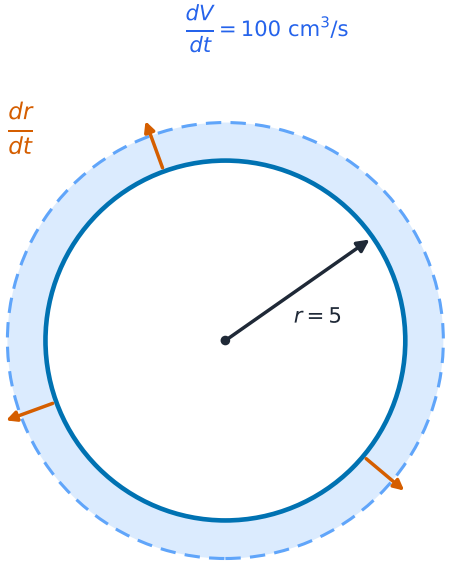
1. **Name the variables** and write down the given rates and the unknown rate (e.g. "$\dfrac{dh}{dt} = -2$ cm/day, find $\dfrac{dV}{dt}$").
2. **Write an equation** relating the quantities (often a **geometric** 几何 or **volume** 体积 formula).
3. **Differentiate both sides with respect to** t (chain rule) – *before* substituting numbers.
4. **Substitute** the known values at the **instant** 时刻 of interest, and solve for the unknown rate.
5. State the answer with **units** and the correct **sign** (a decreasing quantity has a negative rate).

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Worked example. Air is pumped into a spherical balloon so its volume grows at $\dfrac{dV}{dt} = 100$ cm^3/s. How fast is the radius growing when $r = 5$ cm? Start from $V = \frac{4}{3}\pi r^3$ and differentiate with respect to t **first**:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \;\Rightarrow\; 100 = 4\pi(5)^2 \frac{dr}{dt} \;\Rightarrow\; \frac{dr}{dt} = \frac{100}{100\pi} = \frac{1}{\pi} \approx 0.32 \text{ cm/s}.$$

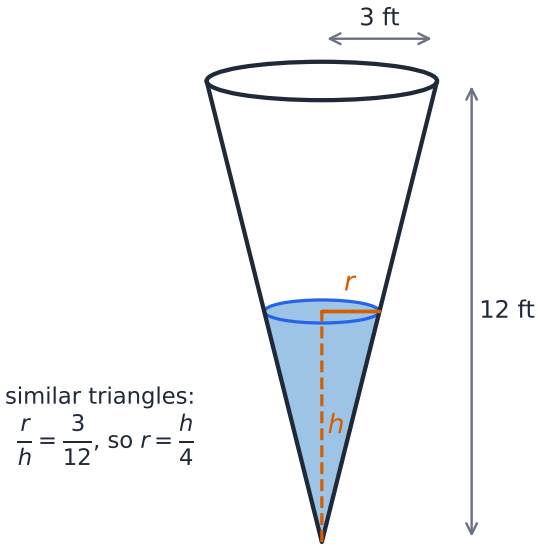
Substituting $r = 5$ *too early* is the classic error – differentiate the general relationship, then plug in.



An inflating balloon links dV/dt and dr/dt through the chain rule

Worked example: conical tank. A tank is a cone with its vertex down, 12 ft deep with a top radius of 3 ft. Water drains out at 2 cubic feet per minute. How fast is the depth falling when the water is 4 ft deep?

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The water forms a smaller cone similar to the tank, so $r = \frac{h}{4}$.

The volume of the water is $V = \frac{1}{3}\pi r^2 h$, which has **two** changing variables. The similar triangles give $\dfrac{r}{h} = \dfrac{3}{12}$, so $r = \dfrac{h}{4}$. Substitute **before** differentiating:

$$V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{48} \quad \Rightarrow \quad \frac{dV}{dt} = \frac{\pi h^2}{16}\,\frac{dh}{dt}.$$

Now substitute the instant: $\dfrac{dV}{dt} = -2$ (draining) and $h = 4$:

$$-2 = \frac{\pi(16)}{16}\,\frac{dh}{dt} \;\Rightarrow\; \frac{dh}{dt} = -\frac{2}{\pi} \approx -0.637 \text{ ft/min}.$$

The depth is falling at about 0.637 feet per minute. Substituting $h = 4$ before differentiating would turn h into a constant and give $\dfrac{dV}{dt} = 0$, which is impossible.

> **Worked example: sliding ladder.** A 10 m ladder leans against a wall. Its foot slides away from the wall at 0.5 m/s. How fast is the top sliding down the wall when the foot is 6 m from the wall?

1. Variables: x = distance from wall to foot, y = height of the top. Given $\dfrac{dx}{dt} = 0.5$ m/s; find $\dfrac{dy}{dt}$ when $x = 6$.
2. Equation (**Pythagoras' theorem** 勾股定理): $x^2 + y^2 = 100$.
3. Differentiate with respect to t: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$.

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4. At this instant $y = \sqrt{100 - 36} = 8$, so $2(6)(0.5) + 2(8)\dfrac{dy}{dt} = 0$, which gives $\dfrac{dy}{dt} = -\dfrac{6}{16} = -0.375$ m/s.
5. The top is sliding **down** at 0.375 m/s (the negative sign means y is decreasing).

Note that $y = 8$ came from the equation at that instant; it is not given in the question.

Common mistakes. Treating a constant as a variable or a variable as a constant: the ladder's length 10 never changes (its derivative is 0), but x and y both do. Dropping the sign: report $\dfrac{dy}{dt} = -0.375$ m/s, or say the top slides **down** at 0.375 m/s, never "down at -0.375 m/s".

Local Linearity and Linearization

Near a **point of tangency** 切点, a **smooth curve** 光滑曲线 looks like its **tangent line** 切线 – this is **local linearity** 局部线性. So the tangent line gives a **linear approximation** 线性近似 (**linearization** 线性化) of the function near that point:

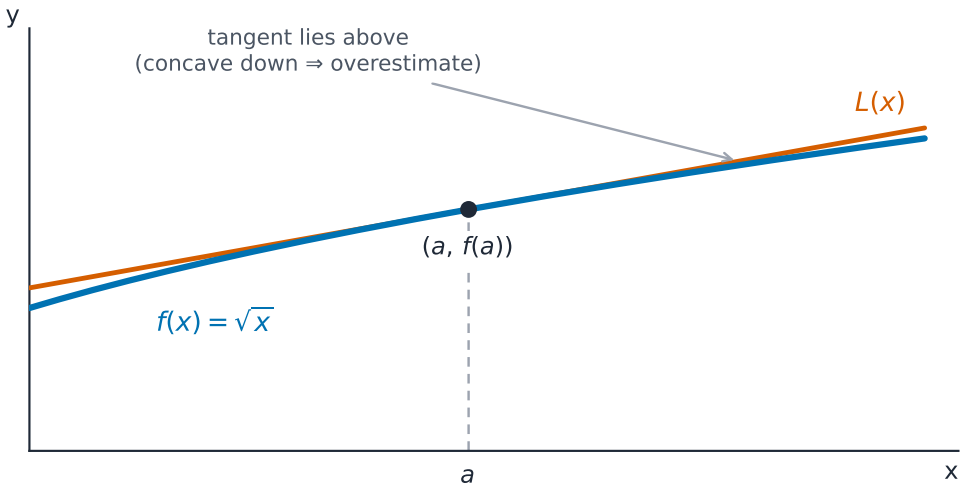
$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

Use it to estimate f at an x close to a.

Over or under? The tangent line lies **below** a concave-up curve and **above** a concave-down curve. So if $f'' > 0$ near a, the linear approximation is an **underestimate**; if $f'' < 0$, an overestimate.

> **Worked example.** $f(2) = 5$, $f'(2) = -1$ and $f''(x) > 0$ for all x. Estimate $f(2.1)$: $L(2.1) = 5 + (-1)(0.1) = 4.9$. Because f is concave up, the estimate 4.9 is an underestimate of $f(2.1)$.

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$$f(x) \approx L(x) = f(a) + f'(a)(x - a) \quad \text{near } x = a$$

The tangent line is the best linear approximation to the curve near the point of tangency

Worked example. Estimate $\sqrt{4.1}$. Take $f(x) = \sqrt{x}$ and $a = 4$: $f(4) = 2$ and $f'(x) = \dfrac{1}{2\sqrt{x}}$ so $f'(4) = \dfrac{1}{4}$. Then $L(4.1) = 2 + \frac{1}{4}(4.1 - 4) = 2.025$ (the true value is $2.0248\ldots$). Because $\sqrt{x}$ is **concave down** 下凹, the tangent lies above the curve, so this is a slight **overestimate** 高估 – justify over/under with the sign of f''.

Common mistake. Using the slope at the new point, $f'(4.1)$, instead of the known point, $f'(4)$. The whole idea is that $f(a)$ and $f'(a)$ are easy to find exactly.

L'Hospital's Rule for Indeterminate Forms

When **direct substitution** 直接代入 in a quotient of limits gives the **indeterminate form** 未定式 $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, you may use **L'Hospital's Rule** 洛必达法则:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided the **numerator** 分子 and **denominator** 分母 are **differentiable** 可导 near the limiting input (excluding the input itself), the denominator derivative is nonzero there, and the derivative-ratio limit exists, finitely or infinitely. Use the corresponding **one-sided** 单侧 conditions when needed. **Differentiate the top and bottom separately** (this is *not* the quotient rule), then try the limit again. First confirm the form really is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ – applying the rule to any other form is a mistake.

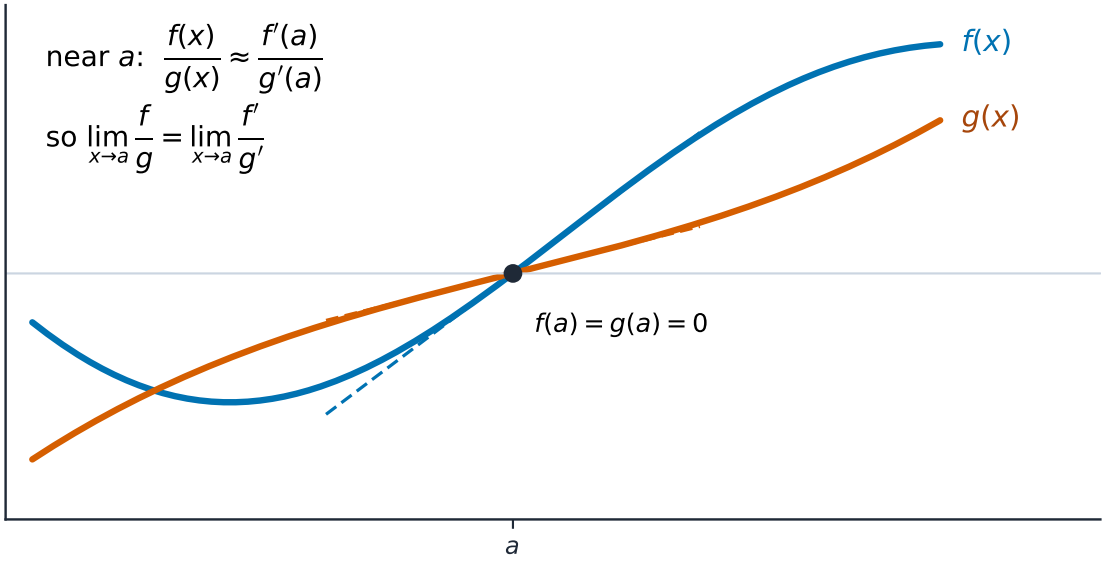
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Worked example. Evaluate $\lim_{x\to 0}\dfrac{\sin x}{x}$. Substituting gives $\frac{0}{0}$, so differentiate top and bottom: $\lim_{x\to 0}\dfrac{\cos x}{1}=\cos 0=1$ – confirming the famous limit from Unit 1.

Worked example. Evaluate $\lim_{x\to\infty}\dfrac{x^2}{e^x}$. Both top and bottom grow without bound: the form is $\frac{\infty}{\infty}$. Apply the rule, check the new form, and apply it again:

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}\ \left(\text{still }\tfrac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{2}{e^x}=0.$$

This illustrates the Unit 1 claim that an **exponential** 指数函数 beats any **polynomial** 多项式. Check the form before **every** use: once it stops being $\frac{0}{0}$ or $\frac{\infty}{\infty}$, stop and substitute.



When $f(a)=g(a)=0$ the ratio f/g near a matches the ratio of their slopes

Worked example (from given values). f is differentiable near 2, with f' continuous at 2, $f(2)=3$ and $f'(2)=8$. Find $\lim_{x\to 2}\dfrac{f(x)-3}{x^2-4}$.

First show the form: f is continuous, so $\lim_{x\to 2}\big(f(x)-3\big)=3-3=0$, and $\lim_{x\to 2}(x^2-4)=0$. The form is $\frac{0}{0}$, so

$$\lim_{x\to 2}\frac{f(x)-3}{x^2-4}=\lim_{x\to 2}\frac{f'(x)}{2x}=\frac{f'(2)}{4}=\frac{8}{4}=2.$$

(The last step needs f' continuous at 2, which exam questions provide; then the derivative-ratio limit exists.) Writing "$=\frac{0}{0}$" as if it were a number, or skipping the

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check of **both** limits, loses the point for L'Hospital's Rule.

Exam tips

- In motion problems: velocity is the derivative of position, acceleration the derivative of velocity; **speed increases when velocity and acceleration share a sign**.
- For **related rates**, differentiate the relating equation with respect to **time**, then substitute the given values last.
- Use the tangent line for a **linear approximation** near a known point; it is accurate only close by.
- Read the sign of a rate: $\frac{dy}{dt} > 0$ means y is increasing at that instant, $\frac{dy}{dt} < 0$ means it is decreasing. A falling height or a shrinking volume has a negative rate.
- Always state units and interpret the answer in context.
- A tangent-line estimate is an underestimate where the curve is concave up, an overestimate where it is concave down.

Practice.

1. $P(t)$ is the number of people in a queue, t in minutes. Interpret $P'(10) = -4$.
2. A particle has $s(t) = t^3 - 9t^2 + 24t$. When is it at rest, and is it speeding up or slowing down at $t = 3$?
3. A circle's radius grows at 0.5 cm/s. How fast is its area growing when $r = 6$ cm?
4. A 1.8 m person walks away from a 6 m lamppost at 1.4 m/s. How fast is the length of their shadow growing?
5. Use linearization to estimate $\sqrt[3]{8.3}$, and say whether it is too big or too small.
6. Evaluate $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Answers.

1. At $t = 10$ minutes, the number of people in the queue is decreasing at 4 people per minute.
2. $v = 3t^2 - 18t + 24 = 3(t-2)(t-4)$: at rest at $t = 2$ and $t = 4$. At $t = 3$, $v = -3 < 0$ and $a = 6t - 18 = 0$, so it is neither speeding up nor slowing down at that instant.
3. $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt} = 2\pi(6)(0.5) = 6\pi \approx 18.8$ cm^2/s.
4. Similar triangles: $\dfrac{s}{1.8} = \dfrac{x+s}{6}$, so $4.2s = 1.8x$ and $s = \frac{3}{7}x$. Then $\dfrac{ds}{dt} = \frac{3}{7}(1.4) = 0.6$ m/s.
5. $f(x) = x^{1/3}$, $a = 8$: $f'(8) = \frac{1}{12}$, so $\sqrt[3]{8.3} \approx 2 + \frac{0.3}{12} = 2.025$. $f'' < 0$ for $x > 0$, so this is too big.
6. $\frac{0}{0}$: $\displaystyle\lim_{x\to 0}\frac{\sin x}{2x}$, still $\frac{0}{0}$: $\displaystyle\lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$.

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Analytical Applications of Differentiation

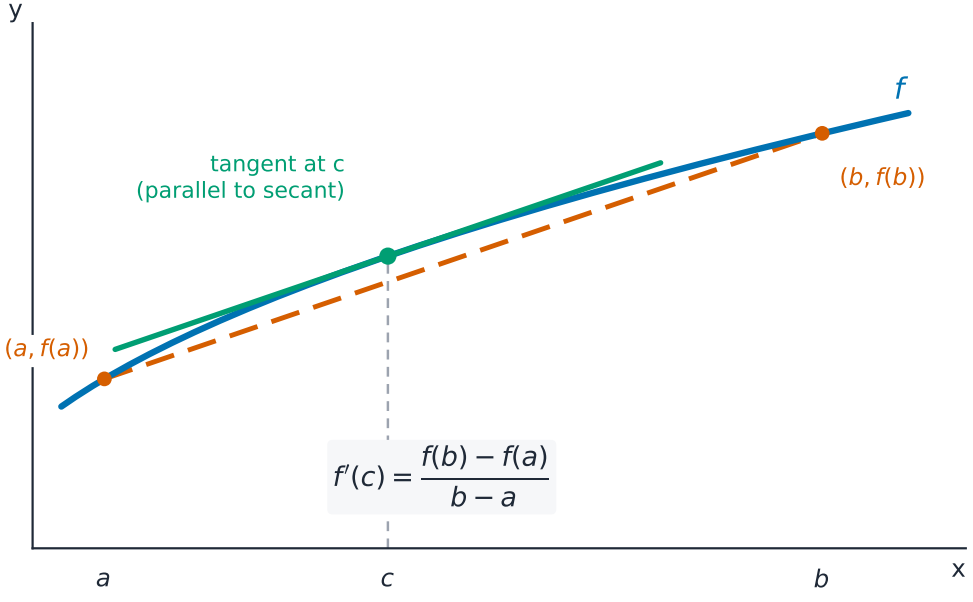
AP Calculus AB

The Mean Value Theorem

The **Mean Value Theorem** 中值定理 (MVT) links the **average rate of change** 平均变化率 to an instantaneous one:

> If f is **continuous** 连续 on $[a, b]$ and **differentiable** 可导 on (a, b), then there is at least one point c in (a, b) where
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In words: somewhere inside the **interval** 区间, the instantaneous rate equals the average rate. Geometrically, some **tangent line** 切线 is **parallel** 平行 to the line joining the **endpoints** 端点.



The Mean Value Theorem: some tangent is parallel to the secant over the interval

Exam skill. Like the IVT, the MVT is an existence theorem, and questions ask you to justify. Full credit needs: (1) state f is continuous on $[a, b]$ and differentiable on (a, b); (2) compute the average rate $\frac{f(b)-f(a)}{b-a}$; (3) conclude "by the MVT there is a c in (a, b) with $f'(c)$ equal to that value." Both hypotheses must be named.

Worked example. Verify the MVT for $f(x) = x^2$ on $[1, 3]$. It is a polynomial, so continuous and differentiable everywhere. The average rate is $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{9 - 1}{2} = 4$. Setting $f'(c) = 2c = 4$ gives $c = 2$, which does lie in $(1, 3)$ – the guaranteed point.

Why both hypotheses matter. Take $f(x) = |x|$ on $[-1, 1]$. It is continuous, and the average rate is $\dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{1 - 1}{2} = 0$. But $f'(x)$ is -1 or 1 everywhere it exists, never 0. The MVT does not apply, because f is not differentiable at $x = 0$, which is inside $(-1, 1)$.

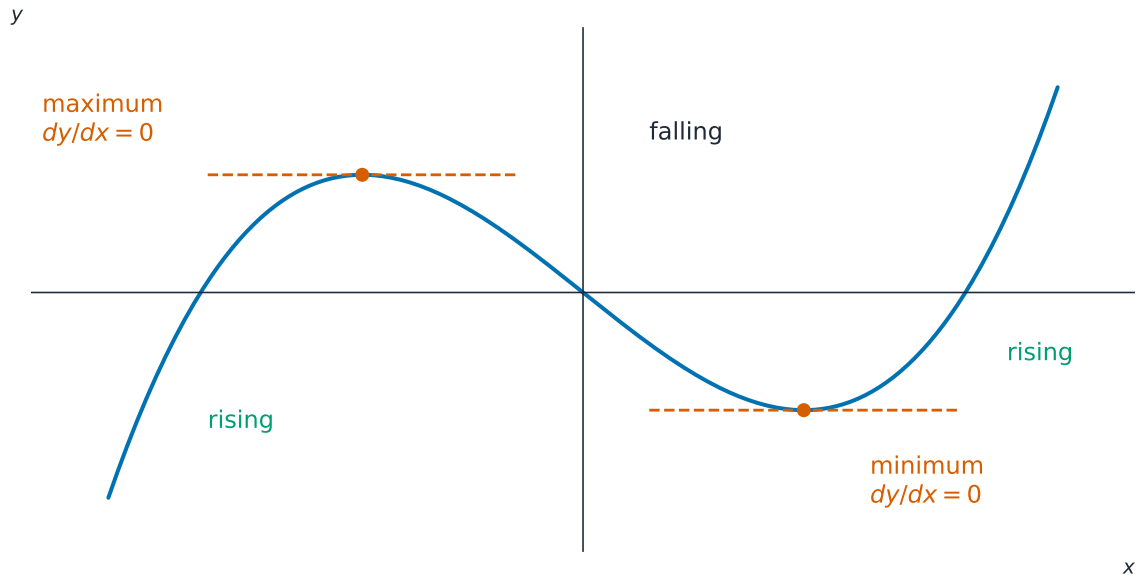
Extreme Values and Critical Points



A mountain peak reflected in a lake: extrema and critical points mark highest and lowest values

The **Extreme Value Theorem** 极值定理 (EVT) guarantees extremes exist: a function **continuous** on a **closed interval** 闭区间 $[a, b]$ attains **both** an **absolute maximum** 最大值 and an **absolute minimum** 最小值 on it.

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At a differentiable interior extremum the derivative is zero

A **critical point** 临界点 is an **interior point** 内点 of the function's **domain** 定义域 where $f'(x) = 0$ **or** $f'(x)$ does not exist. All interior **local (relative) extrema** 局部极值 occur at critical points – but **not** every critical point is an **extremum** 极值. So critical points are the **candidates** 候选点; you must test each.

> **Worked example (a derivative that does not exist).** Find the critical points of $f(x) = x^{2/3}$. Here $f'(x) = \dfrac{2}{3x^{1/3}}$. It is never 0, but it is undefined at $x = 0$, which is in the domain, so $x = 0$ is a critical point. For $x < 0$, $f'(x) < 0$; for $x > 0$, $f'(x) > 0$, so f has a local minimum at $(0, 0)$, a sharp point (a cusp) where no tangent slope exists. Solving only $f'(x) = 0$ would have missed it.

Increasing and Decreasing Intervals

The **first derivative** 一阶导数 tells you where f rises or falls:

- $f'(x) > 0$ on an interval $\Rightarrow$ f is **increasing** 递增 there;
- $f'(x) < 0 \Rightarrow$ f is **decreasing** 递减.

On the exam, "find the intervals where f is increasing" means: find the critical points, then test the sign of f' between them, and **justify with the sign of** f' (a stated reason, not just an interval).

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The First Derivative Test

For f continuous at a critical point $x = c$, classify it as a local max, local min, or neither by checking how f' changes sign there:

- f' changes $+$ **to** $-$ at $c \Rightarrow$ **local maximum** 极大值;
- f' changes $-$ **to** $+$ at $c \Rightarrow$ **local minimum** 极小值;
- f' does **not** change sign $\Rightarrow$ neither.

Always state the **sign change** 符号变化 as your **justification** 理由.

> **Worked example.** Classify the critical points of $f(x) = x^3 - 3x^2$. Here $f'(x) = 3x^2 - 6x = 3x(x-2)$, zero at $x = 0$ and $x = 2$. Testing signs: $f' > 0$ for $x < 0$, $f' < 0$ on $(0, 2)$, and $f' > 0$ for $x > 2$. So f' turns $+ \to -$ at $x = 0$ (a **local maximum**, $f(0) = 0$) and $- \to +$ at $x = 2$ (a **local minimum**, $f(2) = -4$).

Common mistakes. Justifying with "$f'(0) = 0$" alone: a zero of f' only makes $x = 0$ a candidate; the reason is the **sign change** of f'. Testing the sign of f instead of f'. Giving the x-value when the question asks for the maximum **value** ($f(0) = 0$), or the reverse.

The Candidates Test for Absolute Extrema

On a **closed interval**, absolute (global) extrema occur only at **critical points or endpoints**. The **candidates test** 候选点判别法:

1. List all critical points in $[a, b]$ **and** the two endpoints.
2. Evaluate f at each candidate.
3. The largest output is the absolute maximum; the smallest is the absolute minimum.

Show the table of values – the comparison is the argument.

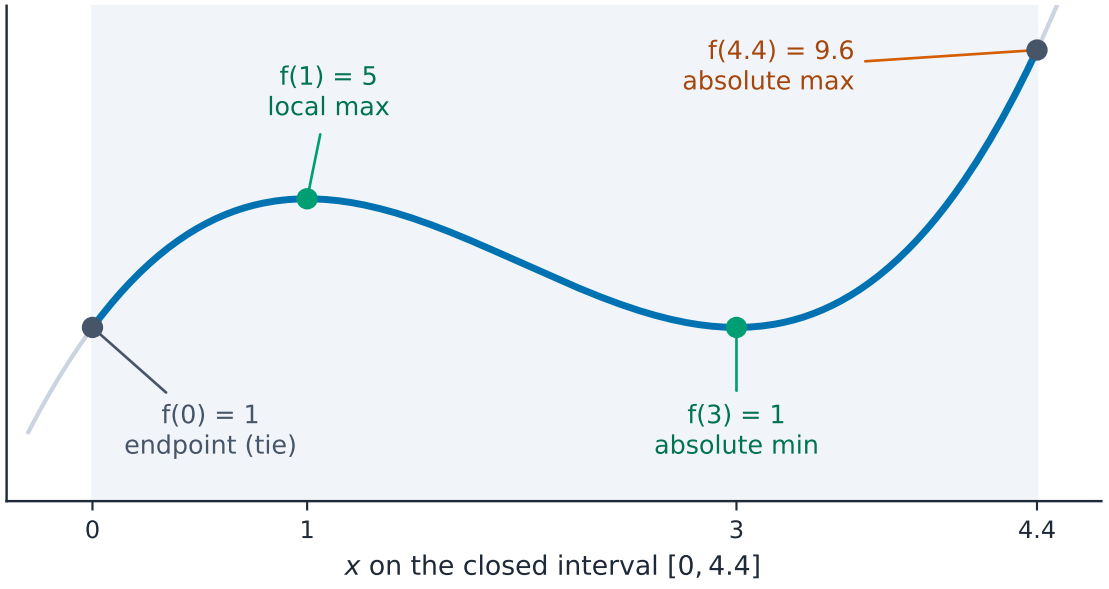
> **Worked example.** Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2$ on $[1, 4]$.

$f'(x) = 3x(x-2)$ is zero at $x = 0$ and $x = 2$. Only $x = 2$ lies in $[1, 4]$, so the candidates are $x = 1$, $x = 2$ and $x = 4$:

x	1 **(endpoint)**	2 **(critical point)**	4 **(endpoint)**
$f(x)$	$1 - 3 = -2$	$8 - 12 = -4$	$64 - 48 = 16$

The absolute maximum is 16, at $x = 4$; the absolute minimum is -4, at $x = 2$. The maximum is at an endpoint where $f'(4) = 24 \neq 0$: this is why the endpoints must be on the list.

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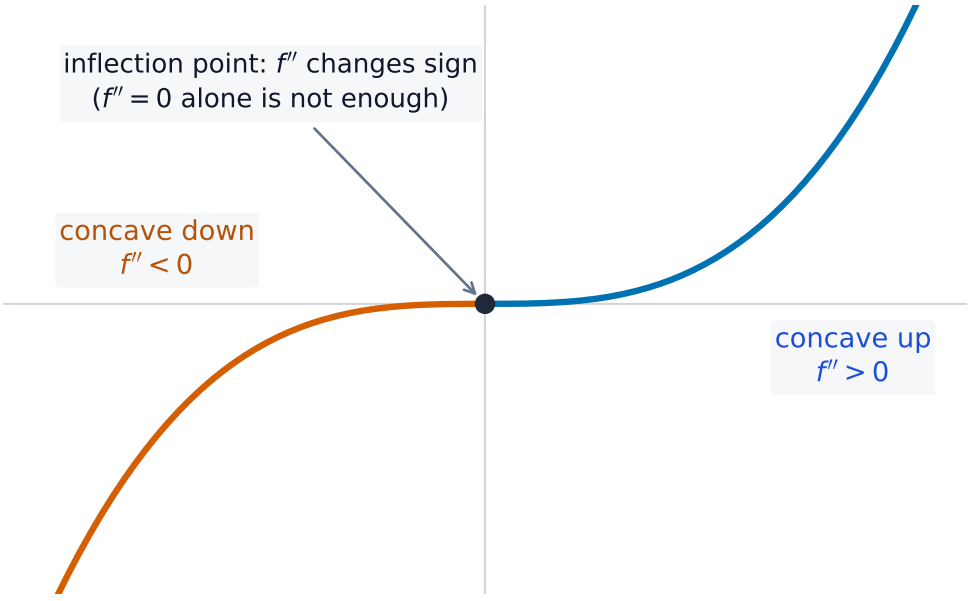
Candidates test: compare f at the critical points and both endpoints

Concavity and Points of Inflection

The **second derivative** 二阶导数 describes bending:

- $f'' > 0 \Rightarrow f$ is **concave up** 上凹 (curving like a cup; f' is increasing);
- $f'' < 0 \Rightarrow f$ is **concave down** 下凹 (f' is decreasing).

A **point of inflection** 拐点 is a continuous point where concavity **changes**. Where the second derivative exists on neighbouring intervals, use its sign change; a zero alone is insufficient. Report its x-coordinate and justify with the sign change of f''.



Concavity comes from the sign of the second derivative; it flips at an inflection point

■ **Worked counterexample: a zero is only a candidate**

For $h(x) = x^4$, $h''(x) = 12x^2$ is positive on both sides of 0, so there is no inflection. Yet $h(x) \geq h(0) = 0$, so 0 is a minimum even though $h''(0) = 0$. For $k(x) = x^3$, $k''(x) = 6x$ changes from negative to positive, giving an inflection at 0. Check neighbouring concavity; the second derivative need not exist at the point itself. The function must be continuous at an inflection point.

The Second Derivative Test

An alternative way to classify a critical point c where $f'(c) = 0$:

- $f''(c) > 0 \Rightarrow$ concave up $\Rightarrow$ **local minimum**;
- $f''(c) < 0 \Rightarrow$ concave down $\Rightarrow$ **local maximum**;
- $f''(c) = 0 \Rightarrow$ the test is **inconclusive** 无法判定 – fall back on the **first derivative test** 一阶导数判别法.

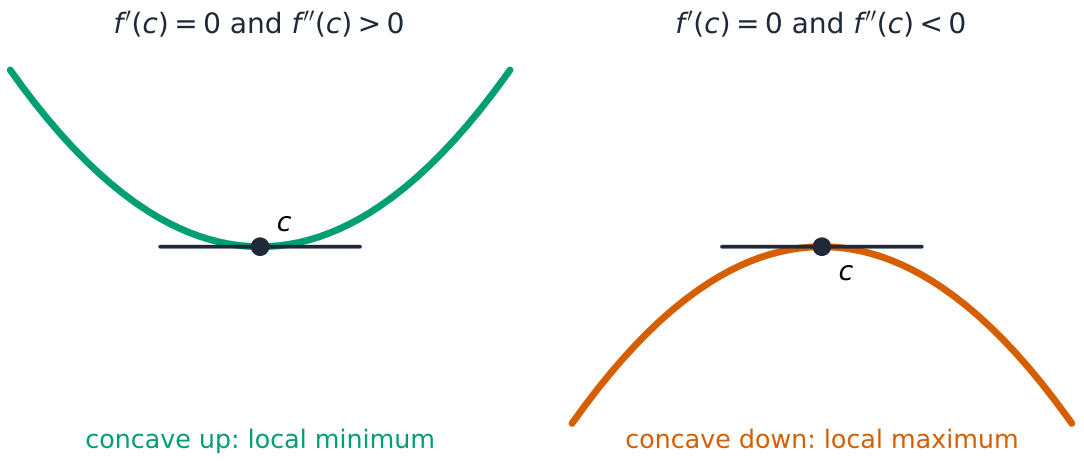
> **Worked example.** For $f(x) = x^3 - 3x^2$, $f'(x) = 3x(x-2)$ is zero at $x = 0$ and $x = 2$, and $f''(x) = 6x - 6$.

- $f''(0) = -6 < 0$, so the graph is concave down there: **local maximum** at $(0, 0)$.
- $f''(2) = 6 > 0$, so it is concave up: **local minimum** at $(2, -4)$.

This agrees with the first derivative test above. For $g(x) = x^4$, however, $g'(0) = 0$ and $g''(0) = 0$: the second derivative test says nothing, and the first derivative test ($g' < 0$ to the left, $g' > 0$ to the right) shows a minimum.

Special case: if a continuous function has only **one** critical point on an interval and it

is a local extremum, that point is also the **absolute** extremum there.

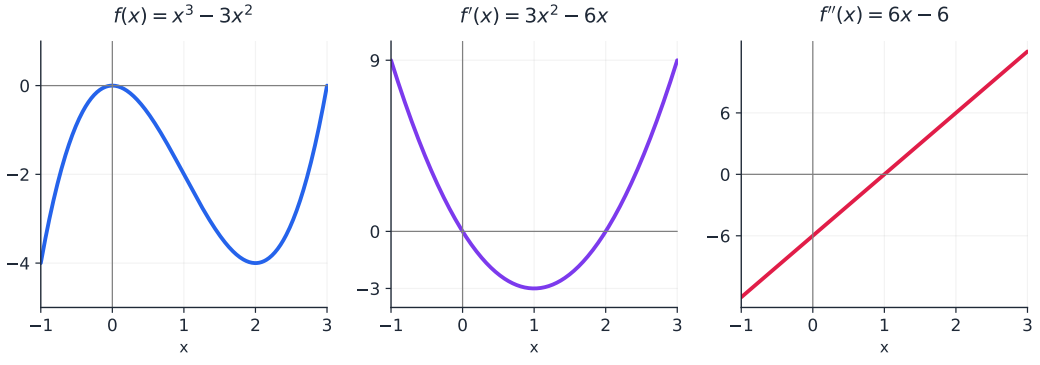


At a critical point $f' > 0$ gives a local minimum and $f' < 0$ a local maximum

Sketching a Function and Its Derivatives

■ Worked example: connect the same three graphs

For $f(x) = x^3 - 3x^2$, the derivative is $f'(x) = 3x(x-2)$ and the second derivative is $f''(x) = 6x - 6$.



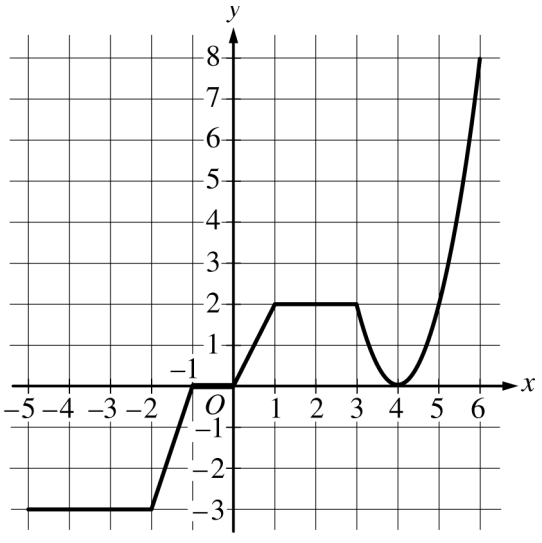
The same cubic and its first and second derivatives, with shared x coordinates.

On $(-1, 0)$, $(0, 2)$ and $(2, 3)$, the signs of f' are positive, negative and positive. Hence f increases, decreases and increases. The sign changes give a local maximum at $(0, 0)$ and minimum at $(2, -4)$. On $(-1, 0)$ the function is increasing but concave down: $f' > 0$ while f' decreases. Concavity changes at 1 because f'' changes from negative to positive; the inflection is $(1, -2)$, where $f'(1) = -3$.

Adding 5 to f leaves both derivatives unchanged. The shape and turning-point x coordinates stay the same; the vertical position needs a given function value.

Connecting f, f-prime, and f-double-prime

■ Original 2018 AB Q 3(c–d): read the displayed derivative

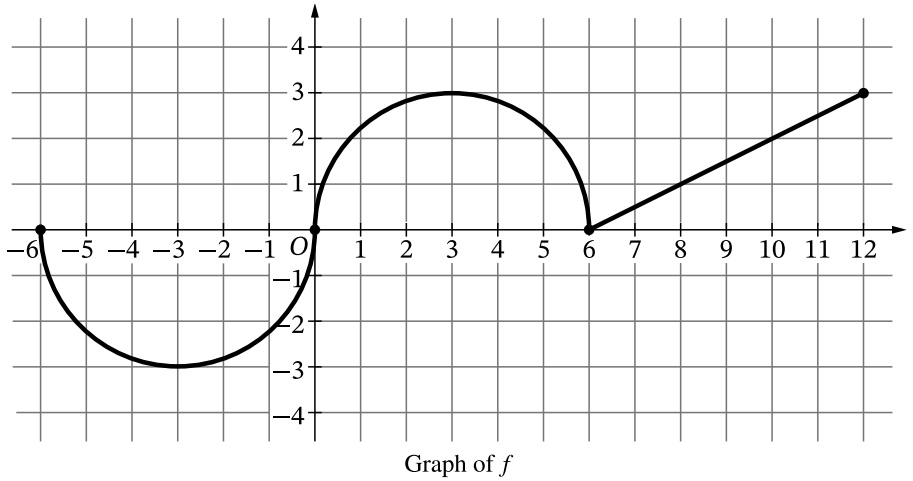


Graph of g

Original graph g=f-prime from 2018 AB Q3, PDF page 4.

The given g is f'. For f to be increasing and concave up, g must be positive and increasing: $(0, 1)$ and $(4, 6)$. At 4, g changes from decreasing to increasing, so f has an inflection. Since g is positive on both sides of 4, f has no extremum there. The official scheme awards both the interval answer and its reason.

■ Original 2025 AB Q 4B: the notation changes



Graph of f

Original graph f=g-prime from 2025 AB Q4, PDF page 7.

Here $g(x) = \int_6^x f(t)\,dt$, so $g' = f$. The inflections of g occur at $x = -3, 3, 6$: the

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displayed f changes decreasing/increasing behaviour there. At 0, f increases on both sides, so g remains concave up. At 6 the change of concavity is valid even though f' is undefined at its corner. The official scheme requires evidence tied to the given graph, not simply naming a change in concavity.

Common mistakes when the graph shown is f'. Reading it as if it were f. On a graph of f': f increases where the graph lies **above** the axis (not where it rises); f has a local extremum where the graph **crosses** the axis; and f has an inflection point where the graph of f' turns (changes from increasing to decreasing, or the reverse).

Introduction to Optimization

Optimization 最优化 uses the derivative to find the largest or smallest value of a quantity on an interval. It is the candidates/derivative-test machinery applied to a real goal.



Fenced enclosures: optimization finds the dimensions that maximise area for a fixed fence length

Solving Optimization Problems

A dependable procedure:

1. Write the quantity to optimize as a function of **one** variable (use a **constraint** 约束条件 equation to eliminate extras).
2. State the interval of allowed inputs.
3. Find critical points ($f' = 0$ or undefined) and test them (first- or second-derivative test, or candidates test if the interval is closed).
4. Answer the question asked, **with units and interpretation** in context – the maximum *area*, the minimum *cost*, etc.

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Worked example. A farmer has 100 m of fence for a rectangular pen against a wall, so only **three** sides need fencing. Maximize the area. Let the two ends be x and the side parallel to the wall be y; the constraint is $2x+y=100$, so $y=100-2x$. The area is

$$A(x)=x(100-2x)=100x-2x^2, \qquad A'(x)=100-4x=0 \;\Rightarrow\; x=25.$$

Since $A''=-4<0$ this is a maximum, giving $y=50$ and $A=25\times 50=1250$ m^2.

Common mistakes. Trying to optimize a function of two variables without using the constraint to remove one; skipping the justification that $x=25$ gives a **maximum** (here $A''<0$); and stopping at $x=25$ when the question asks for the largest **area**.

Behaviors of Implicit Relations

All of this extends to **implicitly defined** 隐式定义 relations. For $\dfrac{dy}{dx}=N/D$, $N=0$ with $D\neq 0$ gives a **horizontal tangent** 水平切线, while $D=0$ with $N\neq 0$ makes the derivative **undefined** 无定义 and gives a **vertical tangent** 竖直切线. If both vanish, further analysis is needed. Because $\dfrac{dy}{dx}$ is usually a relation in x **and** y, and the second derivative involves x, y, and $\dfrac{dy}{dx}$, substitute your first-derivative expression back in when finding $\dfrac{d^2y}{dx^2}$, then reason about concavity from its sign.

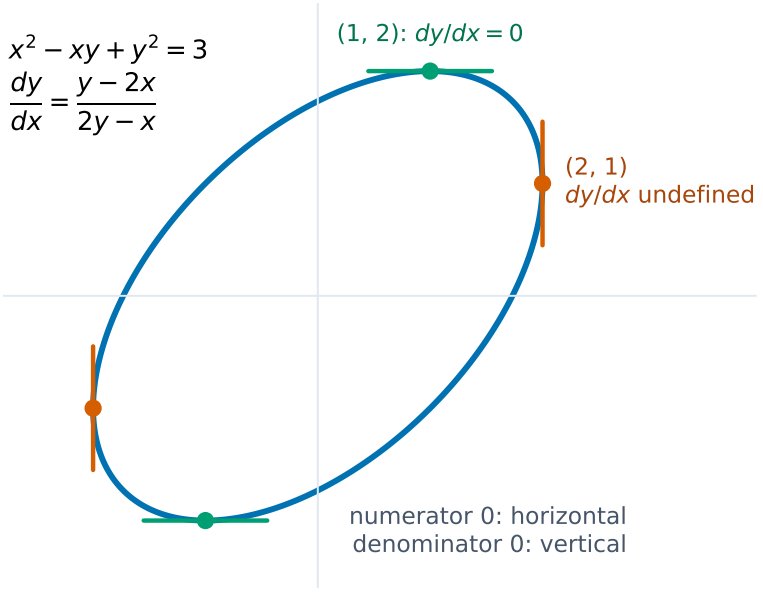
Worked example. The curve $x^2+xy+y^2=27$. Differentiating implicitly, $2x+y+x\dfrac{dy}{dx}+2y\dfrac{dy}{dx}=0$, so

$$\frac{dy}{dx}=-\frac{2x+y}{x+2y}.$$

- **Horizontal tangents:** the numerator is 0, so $y=-2x$. Substituting, $x^2-2x^2+4x^2=3x^2=27$, so $x=\pm 3$: the points $(3,-6)$ and $(-3,6)$.
- **Vertical tangents:** the denominator is 0, so $x=-2y$. Then $4y^2-2y^2+y^2=3y^2=27$, so $y=\pm 3$: the points $(-6,3)$ and $(6,-3)$.
- **Concavity at a horizontal tangent.** Differentiating dy/dx by the quotient rule and using $dy/dx=0$ and $2x+y=0$ at these points leaves $d^2y/dx^2=-2/(x+2y)$. At $(3,-6)$, $x+2y=-9$, so $d^2y/dx^2=2/9>0$: concave up, so $y=-6$ is a local minimum of the curve. At $(-3,6)$ it is $-2/9<0$: a local maximum, $y=6$.

Always check that a point you find lies on the curve: $3^2+3(-6)+(-6)^2=9-18+36=27$.

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Horizontal tangents where $dy/dx = 0$; vertical tangents where it is undefined

Practice

Answer each before reading the answers below.

1. Find the value of c guaranteed by the Mean Value Theorem for $f(x) = x^2$ on $[1, 3]$.
2. For $f(x) = x^3 - 3x^2 - 9x + 5$, find the intervals where f increases and decreases, and the relative extrema. Justify with the first derivative test.
3. Find the absolute maximum and minimum of the function in question 2 on $[0, 4]$.
4. For the same function, find where it is concave up and the point of inflection.
5. $f'(x) = (x - 2)^2(x + 1)$. Find the x-values of all relative extrema of f, and justify your answer.
6. Squares of side x are cut from the corners of a 12 cm by 12 cm sheet, and the sides are folded up to make an open box. Find the greatest volume.
7. For $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$ at $(3, 4)$, and the points where the tangent line is vertical.

Answers

1. $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{9 - 1}{2} = 4 = 2c$, so $c = 2$.
2. $f'(x) = 3(x-3)(x+1)$. Increasing on $x < -1$ and $x > 3$; decreasing on $-1 < x < 3$. Relative maximum $f(-1) = 10$, because f' changes from positive to negative; relative minimum $f(3) = -22$, because f' changes from negative to positive.

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3. Candidates: $f(0) = 5$, $f(3) = -22$, $f(4) = -15$. Absolute maximum 5 at $x = 0$; absolute minimum -22 at $x = 3$.
4. $f''(x) = 6x - 6$: concave up for $x > 1$, down for $x < 1$. Inflection point $(1, -6)$.
5. Only $x = -1$: f' changes from negative to positive there, so f has a relative minimum. At $x = 2$, f' is zero but does not change sign, so there is no extremum.
6. $V = x(12 - 2x)^2$ for $0 < x < 6$. $V' = (12 - 2x)(12 - 6x) = 0$ gives $x = 2$ ($x = 6$ is an endpoint). $V(2) = 2(8)^2 = 128$ cm^3; V is 0 at both ends, so this is the maximum.
7. $2x + 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x}{y} = -\dfrac{3}{4}$. Vertical tangents where $y = 0$: $(5, 0)$ and $(-5, 0)$.

Exam tips

- $f' > 0$ means increasing, $f' < 0$ decreasing; candidates for **extrema** are where $f' = 0$ or is undefined.
- Classify a critical point with the **first-derivative sign change** or the **second-derivative test** 二阶导数判别法 ($f'' > 0$ minimum, $f'' < 0$ maximum).
- $f'' > 0$ is concave up, $f'' < 0$ concave down; a **point of inflection** is where concavity changes (f'' changes sign).
- For an **absolute** extremum on a closed interval, also check the endpoints.
- Justify every conclusion by citing the sign of f' or f'' – the exam demands the reasoning, not just the answer.

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Integration and Accumulation of Change

AP Calculus AB

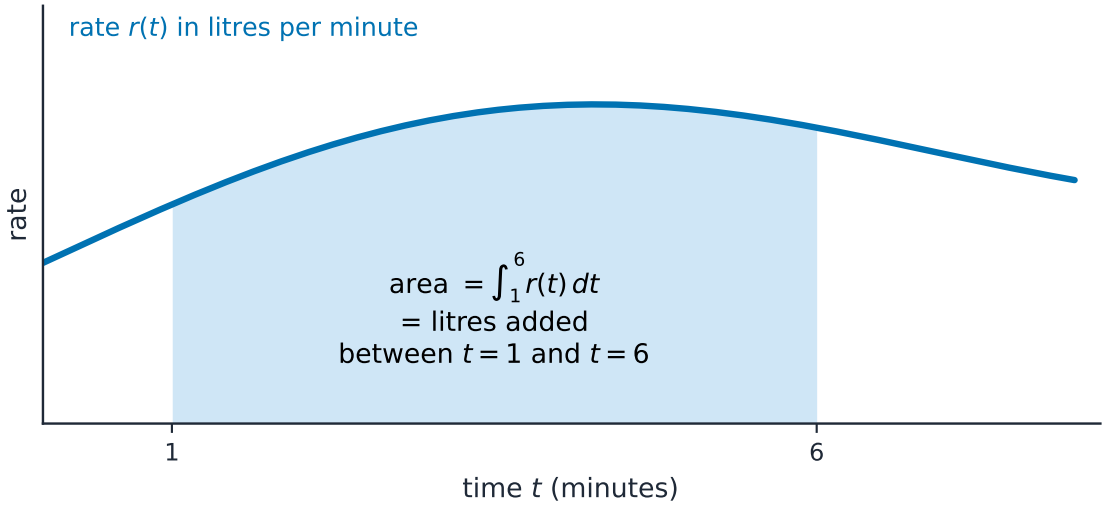
Exploring Accumulations of Change



A water tank: integration accumulates change – total volume from a rate of flow

Differentiation 求导 found rates. **Integration** 积分 runs the idea in reverse: given a **rate of change** 变化率, it finds the **accumulated change** 累积变化. The key picture: the **area** 面积 between the graph of a rate function and the x-axis gives the total accumulation.

- If the rate is positive over an interval, the accumulated change is positive; if negative, negative. Area below the axis counts as negative.
- Simple **regions** 区域 (triangles, **rectangles** 矩形) can be found with **geometry** 几何.
- **Units** 单位: the area's unit is the rate's unit times the input's unit. A rate in vehicles-per-hour times hours gives **vehicles**.

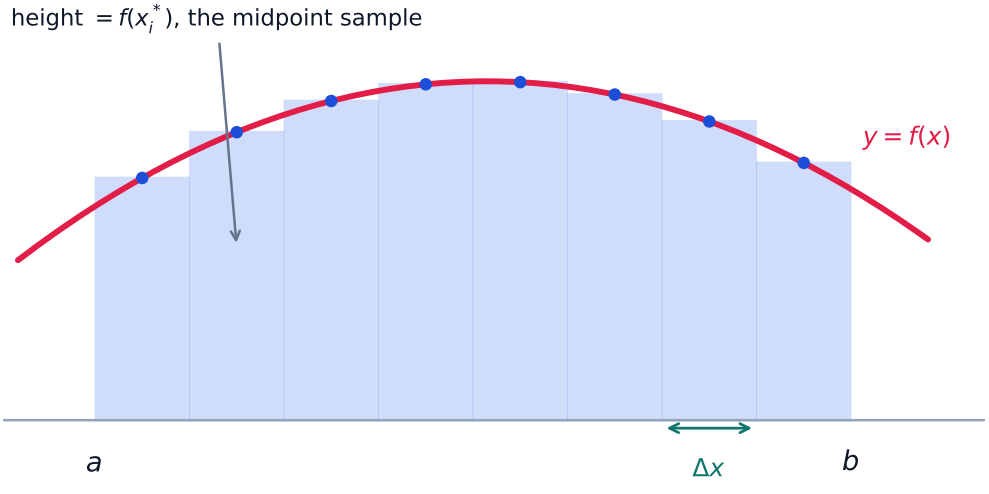


rate × time = amount, so adding thin rectangles gives the total change

The area under a rate graph is the total amount accumulated

Approximating Areas with Riemann Sums

When exact area is hard, approximate it with a **Riemann sum** 黎曼和 – split the interval into **subintervals** 子区间 and add up rectangle (or **trapezoid** 梯形) areas. The four standard estimates:



$$\sum f(x_i^*)\,\Delta x \;\to\; \int_a^b f(x)\,dx \quad \text{as} \quad n \to \infty$$

Each rectangle has area $f(x)\,\Delta x$; adding them estimates the area, and as the strips narrow the sum approaches the definite integral.

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- **Left** Riemann sum – rectangle height from the **left endpoint** 端点 of each subinterval.
- **Right** Riemann sum – height from the **right** endpoint.
- **Midpoint** Riemann sum – height from the **midpoint** 中点.
- **Trapezoidal sum** 梯形法 – average the two endpoint heights (a trapezoid).

Subintervals may be **uniform** 等宽的 (equal width) or **nonuniform** 不等宽的 – read widths from the table.

Over- or underestimate? Judge from the behavior of the function: for an **increasing** function, a left sum underestimates and a right sum overestimates; a **trapezoidal** sum overestimates when the function is **concave up** and underestimates when concave down. Exam parts ask you to state which and why.

Worked example. An increasing function f on $[0,6]$ has table values $f(0)=3$, $f(2)=5$, $f(4)=8$, $f(6)=9$. Estimate $\int_0^6 f(x)\,dx$ with a **right** Riemann sum of three equal subintervals ($\Delta x = 2$). Use the right endpoint of each strip:

$$2\big(f(2)+f(4)+f(6)\big) = 2(5+8+9) = 44.$$

Since f is increasing, this right sum is an **overestimate** 高估; the left sum $2(3+5+8)=32$ would be an **underestimate** 低估.

Worked example: trapezoidal sum, unequal widths. Water flows into a tank at rate $R(t)$ litres per minute.

t **(minutes)**	0	2	5	10
$R(t)$ (litres/min)	4	6	7	3

Estimate $\int_0^{10} R(t)\,dt$ with a trapezoidal sum. The widths are 2, 3 and 5, so each trapezoid needs its own width:

$$\int_0^{10} R(t)\,dt \approx 2\cdot\frac{4+6}{2} + 3\cdot\frac{6+7}{2} + 5\cdot\frac{7+3}{2} = 10 + 19.5 + 25 = 54.5 \text{ litres}.$$

Meaning: about 54.5 litres flow in during the first 10 minutes. A common error is to use one width for every strip when the table steps are unequal.

Riemann Sums and Integral Notation

As the subinterval widths shrink to zero, the Riemann sum approaches an exact value – the **definite integral** 定积分:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

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Here Δx_i is the width of the ith subinterval and x_i^* a point inside it. So a definite integral *is* the **limit** 极限 of a Riemann sum, and you should be able to translate each into the other.

Worked example (limit of a sum to an integral). Write $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(1+\frac{3i}{n}\right)^2$ as a definite integral and evaluate it.

Match the pieces: $\Delta x = \dfrac{3}{n}$, so the interval has width 3; $x_i = 1 + \dfrac{3i}{n}$ runs from 1 to 4; and $f(x) = x^2$. So the limit is

$$\int_1^4 x^2\,dx = \frac{4^3 - 1^3}{3} = 21.$$

The same limit is also $\displaystyle\int_0^3 (1+x)^2\,dx$ (with $x_i = \tfrac{3i}{n}$); both are correct, as long as f and the limits match.

The Fundamental Theorem of Calculus and Accumulation Functions

A definite integral with a variable **upper limit** 上限 defines a new **accumulation function** 累积函数. The **Fundamental Theorem of Calculus** 微积分基本定理 (first part) says differentiation undoes this accumulation: if f is **continuous** 连续, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

So if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$ and $g''(x) = f'(x)$. This is the engine behind the very common "let $g(x) = \int_a^x f(t)\,dt$" questions.

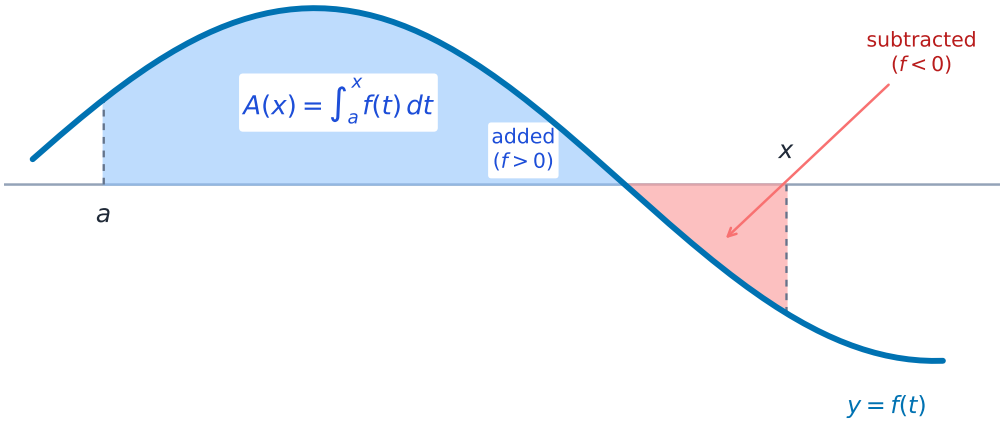
When the upper limit is a function of x, the **chain rule** 链式法则 adds a factor:

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f\big(u(x)\big)\cdot u'(x), \qquad \text{e.g.} \qquad \frac{d}{dx}\int_1^{x^2} \cos t\,dt = \cos(x^2)\cdot 2x.$$

Common mistakes. Forgetting the chain-rule factor $u'(x)$ when the upper limit is not plain x, and differentiating f instead of evaluating it: $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt$ is $f(x)$, not $f'(x)$.

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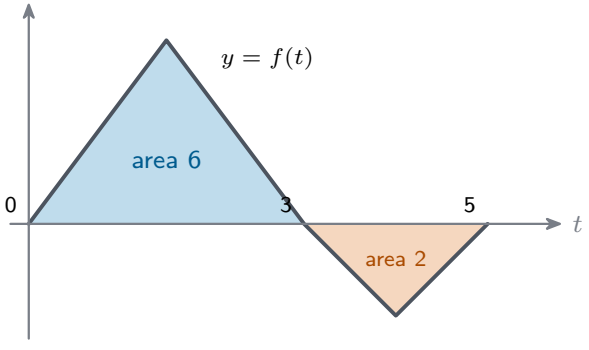
$A'(x) = f(x)$ (Fundamental Theorem)



The accumulation function adds signed area; the FTC says its derivative is f

Behavior of Accumulation Functions

Because $g'(x) = f(x)$, everything from Unit 5 applies to an accumulation function using the graph of f:



$g(3) = 6$: maximum, f goes $+$ to $-$

$g(5) = 6 - 2 = 4$, not 8

Accumulate signed area: subtract what lies below.

- g is **increasing** where $f > 0$ and **decreasing** where $f < 0$;
- g has a **local extremum** 局部极值 where f **crosses zero** (with a **sign change** 符号变化);
- g is **concave up** 上凹 where f is **increasing**; **inflection points** 拐点 of g occur where f has a local extremum.

To get a *value* of g, compute the **signed area** 有符号面积: $g(x) = \int_a^x f(t)\,dt$, adding areas above the axis and subtracting areas below.

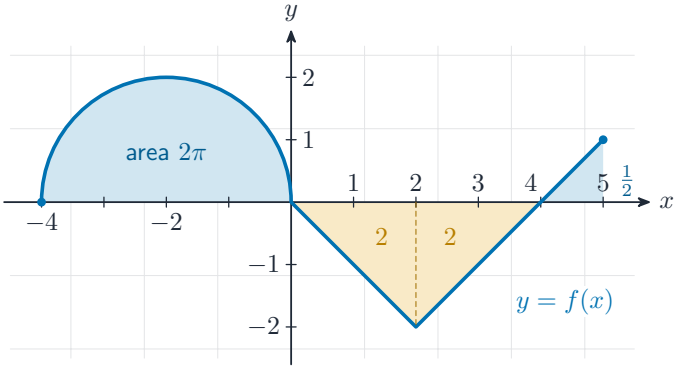
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Common mistake. Adding an area below the axis instead of subtracting it. Suppose f lies above the axis on $[0,3]$ with area 6 and below it on $[3,5]$ with area 2. Then $g(5)=\int_0^5 f(t)\,dt = 6-2=4$, and g has its maximum at $x=3$, where f changes from positive to negative. The total area, 8, answers a different question.

Worked example (going left of the start). Let $g(x)=\displaystyle\int_{-2}^{x} f(t)\,dt$, where the graph of f lies above the axis on $[-4,-2]$ and bounds an area of 3 there. Then $g(-4)=\displaystyle\int_{-2}^{-4} f(t)\,dt = -\int_{-4}^{-2} f(t)\,dt = -3$. Integrating **backwards** over a positive area gives a negative value.

■ Worked example: the graph-of-f question

This is one of the most common free-response questions. The graph of f on $[-4,5]$ is an upper semicircle of radius 2 and three line segments. Let $g(x)=\displaystyle\int_0^x f(t)\,dt$.



The graph of f: a semicircle and three line segments, with the area of each region marked

(a) Find $g(-4)$, $g(2)$ and $g(5)$. Use the areas, with a sign. The semicircle has area $\frac{1}{2}\pi(2)^2=2\pi$, and each triangle below the axis has area $\frac{1}{2}(2)(2)=2$.

- $g(-4)=\displaystyle\int_0^{-4} f(t)\,dt = -\int_{-4}^{0} f(t)\,dt = -2\pi$ (backwards over a positive area).
- $g(2)=-2$ (one triangle below the axis).
- $g(5)=-2-2+\frac{1}{2}=-3.5$.

(b) On which intervals is g increasing? Find the x-values of the relative extrema of g. Since $g'(x)=f(x)$, g is increasing on $[-4,0]$ and $[4,5]$, where $f\ge 0$. g has a **relative maximum** at $x=0$, because $g'=f$ changes from positive to negative there, and a **relative minimum** at $x=4$, because $g'=f$ changes from negative to positive.

(c) Find the x-values of the points of inflection of g. $g''(x)=f'(x)$, so look

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at where f changes from increasing to decreasing, or back. f increases on $(-4,-2)$, decreases on $(-2,2)$ and increases on $(2,5)$, so g has points of inflection at $x=-2$ and $x=2$. ($x=0$ is not one: f is decreasing on both sides of it.)

(d) Find the absolute maximum and minimum values of g on $[-4,5]$. Use the candidates test: check the endpoints and the critical points.

x	$g(x)$	**Why it is a candidate**
-4	$-2\pi \approx -6.28$	endpoint
0	0	critical point: $g'(0)=f(0)=0$
4	-4	critical point: $g'(4)=f(4)=0$
5	-3.5	endpoint

The absolute maximum is 0, at $x=0$; the absolute minimum is -2π, at $x=-4$.

(e) Write an equation of the line tangent to g at $x=2$. $g(2)=-2$ and $g'(2)=f(2)=-2$, so $y=-2-2(x-2)$.

Exam tip. Every justification names $g'=f$ (or $g''=f'$) and says what **changes sign**: "$g'=f$ changes from positive to negative at $x=0$" earns the point; "the graph goes down" does not. Read values of g from areas, never from the height of the graph of f: the height gives g'.

Properties of Definite Integrals

These properties simplify computation and appear constantly:

$$\int_a^a f = 0, \qquad \int_b^a f = -\int_a^b f, \qquad \int_a^b \big(f \pm g\big) = \int_a^b f \pm \int_a^b g,$$

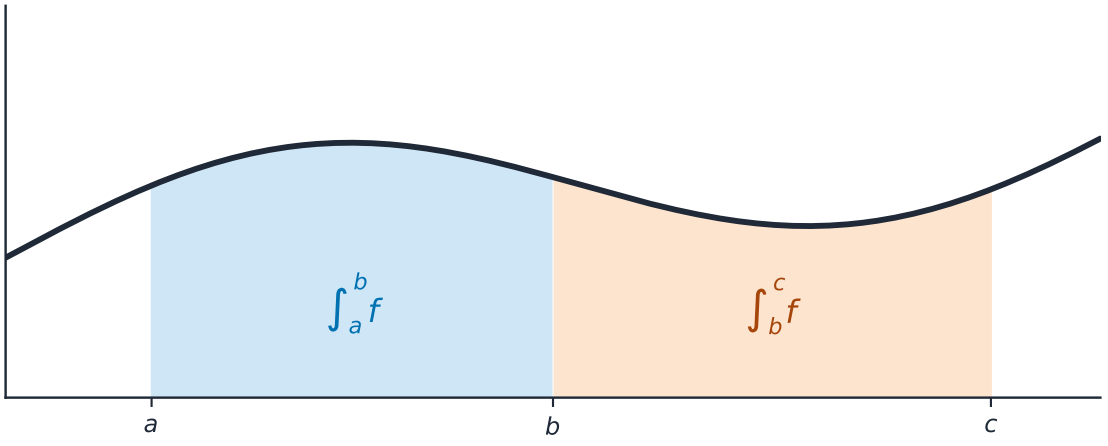
$$\int_a^b k\,f = k\int_a^b f, \qquad \int_a^c f + \int_c^b f = \int_a^b f.$$

The last (splitting at an **interior point** 内点 c) lets you build a total integral from pieces read off a graph.

Worked example. Given $\displaystyle\int_0^5 f(x)\,dx = 7$ and $\displaystyle\int_0^2 f(x)\,dx = 3$, find $\displaystyle\int_5^2 \big(2f(x)+1\big)\,dx$.

Splitting gives $\displaystyle\int_2^5 f = 7-3=4$. Then $\displaystyle\int_2^5 \big(2f(x)+1\big)\,dx = 2(4)+3 = 11$, because the constant 1 integrates to the width 3. Reversing the limits changes the sign, so the answer is -11. Forgetting the width (writing $2(4)+1$) is the common slip.

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$$\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx \qquad \int_b^a f\,dx = -\int_a^b f\,dx$$

Integrals over adjacent intervals add; reversing the limits changes the sign

The Fundamental Theorem and Evaluating Integrals

The second part of the Fundamental Theorem evaluates a definite integral using an **antiderivative** 原函数. If $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

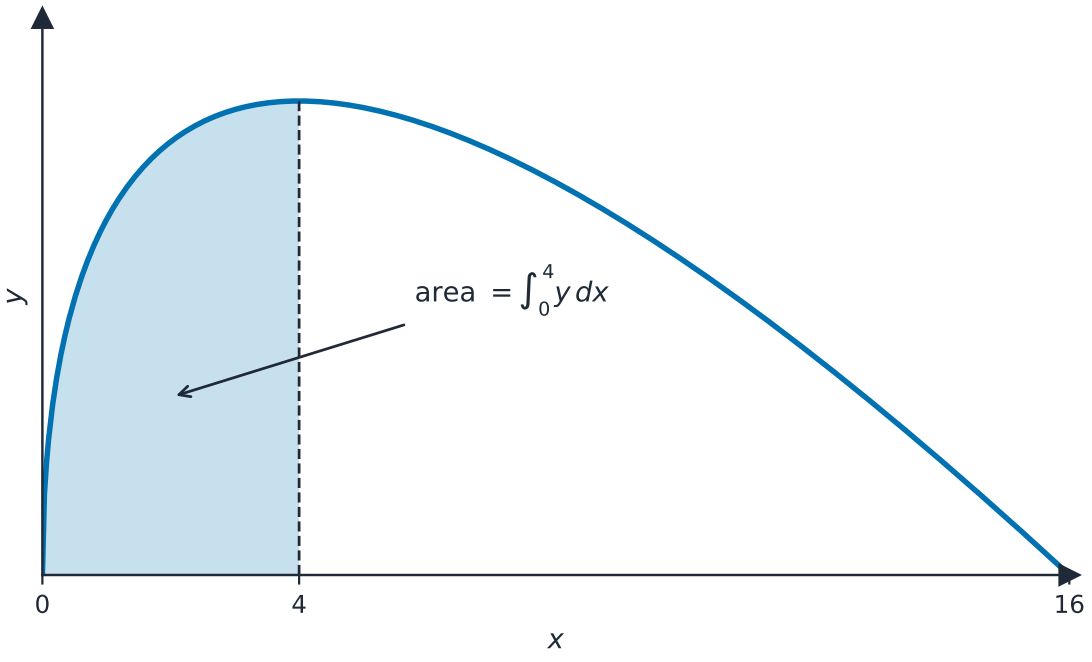
So: find any antiderivative F, then subtract its values at the two limits. This is how most exact integrals are computed. It also gives the **net change** 净变化 view: $\int_a^b g'(t)\,dt = g(b) - g(a)$, so a starting value plus accumulated change gives a later value, e.g. $g(5) = g(0) + \int_0^5 g'(t)\,dt$.

Worked example. Evaluate $\int_1^3 (2x+1)\,dx$. An antiderivative is $F(x) = x^2 + x$, so

$$\int_1^3 (2x+1)\,dx = F(3) - F(1) = (9+3) - (1+1) = 12 - 2 = 10.$$

Common mistake. Dropping the brackets: $9 + 3 - 1 + 1 = 12$ is wrong. Subtract the **whole** of $F(1)$.

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A definite integral is the signed area between the curve and the x-axis

Antiderivatives and Indefinite Integrals

An **indefinite integral** 不定积分 is the family of all antiderivatives, written with a **constant of integration** 积分常数:

$$\int f(x)\,dx = F(x) + C.$$

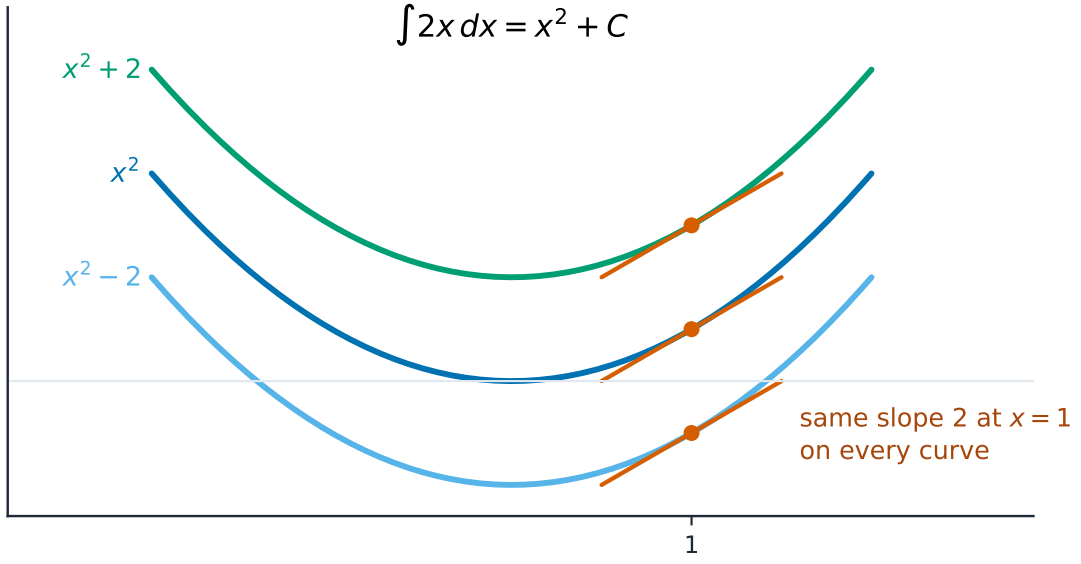
Reverse each derivative rule to build the basic antiderivatives:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1), \qquad \int \frac{1}{x}\,dx = \ln|x| + C, \qquad \int e^x\,dx = e^x + C,$$

$$\int \cos x\,dx = \sin x + C, \qquad \int \sin x\,dx = -\cos x + C.$$

Common mistakes. Leaving out $+C$ on an indefinite integral; losing the minus sign in $\int \sin x\,dx = -\cos x + C$; and using the power rule on x^{-1}, which would divide by 0: $\int \frac{1}{x}\,dx = \ln|x| + C$.

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Antiderivatives differ by a constant: vertical shifts with the same slopes

Integrating Using Substitution

u-substitution 换元积分法 reverses the chain rule. Choose an inside function $u = g(x)$, so $du = g'(x)\,dx$, and rewrite the integral entirely in u:

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du.$$

Look for a function *and its* ***derivative*** 导数 both present. For a **definite** integral, either change the limits to u-values or convert back to x before substituting the original limits.

Worked example. Evaluate $\int 2x\cos(x^2)\,dx$. The inside function is $u = x^2$, whose derivative $2x\,dx = du$ is present, so

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

Spotting that $2x$ is exactly $\dfrac{du}{dx}$ is the whole trick.

Worked example: definite integral. Evaluate $\displaystyle\int_0^1 x\,(x^2+1)^3\,dx$.

Let $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Change the limits: $x = 0$ gives $u = 1$, and $x = 1$ gives $u = 2$. Then

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$$\int_0^1 x\,(x^2+1)^3\,dx = \frac{1}{2}\int_1^2 u^3\,du = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{8}\,(16-1) = \frac{15}{8}.$$

With the new limits there is no need to go back to x. Keeping the old limits 0 and 1 with u is a common error.

Long Division and Completing the Square

Two algebraic set-up moves let more integrals fit the basic forms: **polynomial long division** 多项式长除法 when the top **degree** 次数 is $\geq$ the bottom degree of a **rational function** 有理函数, and **completing the square** 配方法 to turn a **quadratic** 二次 **denominator** 分母 into a form that integrates to an **arctangent** 反正切 or **logarithm** 对数.

Worked example: long division. Find $\displaystyle\int \frac{x^2+1}{x+1}\,dx$. The top degree (2) is at least the bottom degree (1), so divide first: $x^2+1 = (x+1)(x-1)+2$. Then

$$\int \frac{x^2+1}{x+1}\,dx = \int \left(x-1+\frac{2}{x+1}\right)dx = \frac{x^2}{2} - x + 2\ln|x+1| + C.$$

Worked example: completing the square. Find $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$. Complete the square: $x^2+2x+5 = (x+1)^2+4$. With $u = x+1$ and $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a}+C$ (here $a=2$):

$$\int \frac{1}{(x+1)^2+4}\,dx = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C.$$

Check any answer like this by differentiating it back.

Selecting Techniques for Antidifferentiation

A skill topic: match the integral to a method. Try a basic antiderivative first; look for a u-substitution (an inside function whose derivative is also present); use algebra (division, completing the square, splitting a fraction) to reshape the **integrand** 被积函数 into a standard form. Naming the structure first prevents wasted effort.

Worked example (three look-alikes).

- $\displaystyle\int \frac{x}{x^2+1}\,dx$: the derivative of the denominator is almost on top, so substitute $u = x^2+1$: $\frac{1}{2}\ln(x^2+1)+C$.
- $\displaystyle\int \frac{x^2+1}{x}\,dx$: split the fraction: $\displaystyle\int \left(x+\frac{1}{x}\right)dx = \frac{x^2}{2} + \ln|x| + C$.

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- $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$: complete the square, $x^2+4x+5=(x+2)^2+1$: $\arctan(x+2)+C$.

Exam tips

- For $n \neq -1$, use $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ on an interval where the **power rule** 幂法则 applies. For $n=-1$, use $\ln|x|+C$. Do not forget $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f'(x)\,dx = f(b)-f(a)$, and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.
- A definite integral is a signed area (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.

Practice.

1. Write $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{\frac{2i}{n}}$ as a definite integral.
2. Find $\displaystyle\frac{d}{dx}\int_1^{3x} e^{-t^2}\,dt$.
3. Given $\displaystyle\int_1^4 g(x)\,dx = 10$, find $\displaystyle\int_4^1 \big(3-g(x)\big)\,dx$.
4. Evaluate $\displaystyle\int_0^{\pi/2} \sin x\cos^2 x\,dx$.
5. Find $\displaystyle\int \frac{e^x}{1+e^x}\,dx$.
6. A left Riemann sum with 3 equal subintervals for $\displaystyle\int_0^6 f(x)\,dx$, where $f(0)=4$, $f(2)=7$, $f(4)=6$, $f(6)=1$.

Answers.

1. $\displaystyle\int_0^2 \sqrt{x}\,dx$.
2. $e^{-9x^2}\cdot 3 = 3e^{-9x^2}$.
3. $\displaystyle\int_1^4 (3-g)\,dx = 9-10=-1$, so the reversed integral is 1.
4. $u=\cos x$, $du=-\sin x\,dx$: $\displaystyle\int_1^0 -u^2\,du = \int_0^1 u^2\,du = \frac{1}{3}$.
5. $\ln(1+e^x)+C$.
6. $2(4+7+6)=34$.

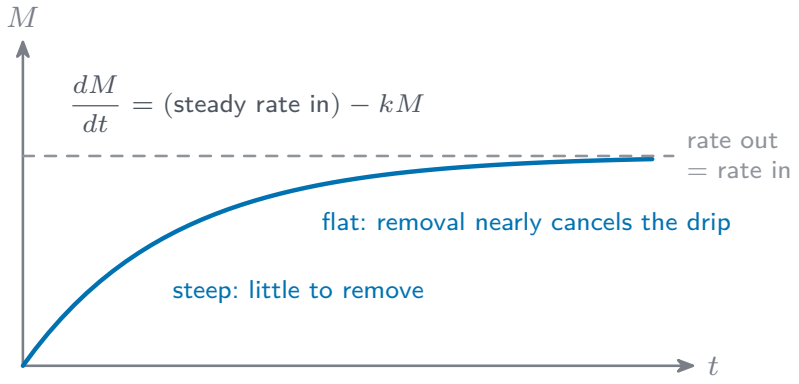
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Differential Equations

AP Calculus AB

Modeling Situations with Differential Equations

A nurse sets up a drip. The medicine flows into the blood at a steady rate, and the body removes it at a rate that grows with the amount already there. Nobody can write a formula for the amount $M(t)$ straight away, but everyone can say **how fast it changes**. That sentence is a differential equation, and this unit teaches you to write one, read its picture and solve it.



Read the picture from the rate.

A **differential equation** 微分方程 is an equation that relates a function to its own **derivatives** 导数. It describes a situation by its **rate of change** 变化率. For example, "the rate of change of a quantity is **proportional** 成正比 to its size" becomes

$$\frac{dy}{dt} = ky.$$

Learning to translate a sentence about a rate into a differential equation is the first skill of this unit.

> **Worked example: from words to an equation.** Read "is proportional to" as "equals a constant k times", and "is inversely proportional to" as "equals a constant k divided by".

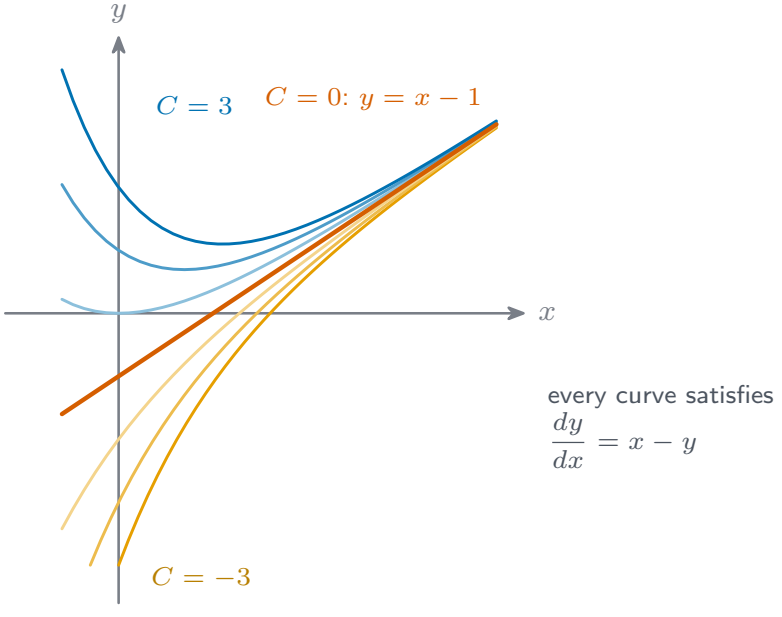
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sentence	differential equation
A population P grows at a rate proportional to its size.	$\dfrac{dP}{dt} = kP$
The temperature T of a drink changes at a rate proportional to the difference between T and the room temperature 20°C.	$\dfrac{dT}{dt} = k(T - 20)$
The depth h of water in a tank decreases at a rate proportional to $\sqrt{h}$.	$\dfrac{dh}{dt} = -k\sqrt{h}$, with $k > 0$
The volume V changes at a rate inversely proportional to V^2.	$\dfrac{dV}{dt} = \dfrac{k}{V^2}$

The word "decreases" gives the minus sign; with it written explicitly, k stays positive.

Verifying Solutions for Differential Equations

A **solution** 解 of a differential equation is a function that makes it true. You can **verify** 验证 a proposed solution by differentiating it and substituting into the equation: if both sides match, it is a solution. Note a differential equation usually has **infinitely many** solutions – a whole **family of solutions** 解族 – indexed by an **arbitrary constant** 任意常数, which need not be an additive constant.



One differential equation, a whole family of solutions.

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Worked example: verifying a family. Show that $y = Ce^{-x} + x - 1$ is a solution of $\dfrac{dy}{dx} = x - y$ for every constant C.

Left side: $\dfrac{dy}{dx} = -Ce^{-x} + 1$. Right side: $x - y = x - (Ce^{-x} + x - 1) = 1 - Ce^{-x}$. The two sides are equal for all x, so every member of the family is a solution. (With $C = 0$ the solution is the straight line $y = x - 1$.)

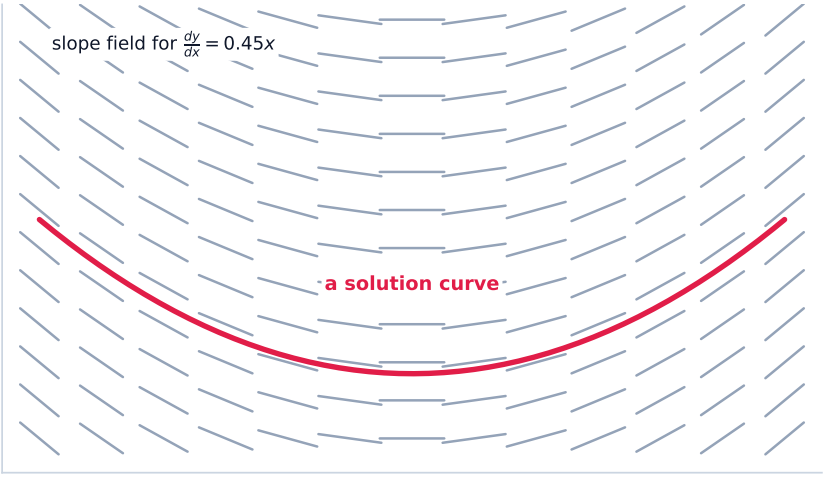
Worked example: an unknown linear solution. For what values of m and b is the line $y = mx + b$ a solution of $\dfrac{dy}{dx} = 2x - y$?

Substitute: the left side is m; the right side is $2x - (mx + b) = (2 - m)x - b$. For $m = (2-m)x - b$ to hold for **every** x, the x-terms must match ($0 = 2 - m$, so $m = 2$) and the constants must match ($m = -b$, so $b = -2$). The solution is $y = 2x - 2$. Check: $\dfrac{dy}{dx} = 2$ and $2x - (2x - 2) = 2$.

Common mistake. Substituting a single point and stopping. A solution must make the equation true at **every** x in its domain, not at one point.

Sketching Slope Fields

A **slope field** 斜率场 draws the differential equation as a grid of short segments: at each point (x, y) the segment has slope $\dfrac{dy}{dx}$ evaluated there. To sketch one, plug several points into the right-hand side and draw a small segment with that slope at each. The picture shows the shape of the **solution curves** 解曲线 without solving.



each segment is the local slope $\frac{dy}{dx}$; a solution curve is tangent to every segment it meets

Each short segment has the slope $\dfrac{dy}{dx}$ at that point; a solution curve threads through, staying tangent to the field.

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Exam skill. A common part gives a portion of a slope field and asks you to sketch the particular solution through a given point – start at the point and follow the segments, staying **tangent** 相切 to them.

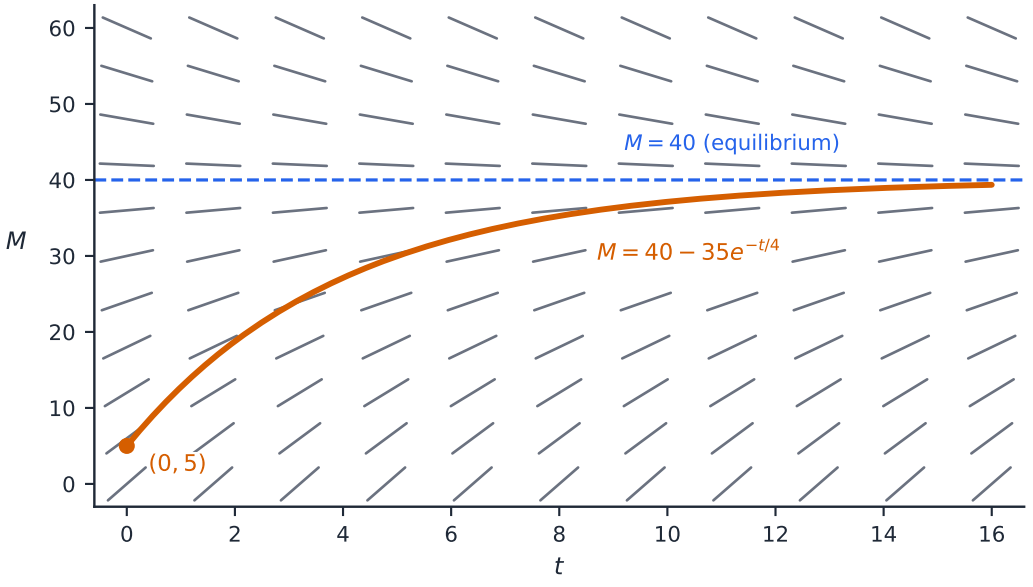
> **Worked example.** Sketch the slope field of $\dfrac{dy}{dx} = x - y$ at a few points. Substitute each point into $x - y$:

point (x, y)	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(-1,1)$
slope $x - y$	0	1	-1	0	1	-2

Draw a flat segment at $(0,0)$ and $(1,1)$, a segment rising at $45°$ at $(1,0)$ and $(2,1)$, and so on. Every point on the line $y = x$ has slope 0, and every point on $y = x - 1$ has slope 1.

Reasoning Using Slope Fields

Solutions to a differential equation are **functions or families of functions**. Read a slope field to reason about behavior: where the segments are flat ($\frac{dy}{dx} = 0$) the solution is momentarily level; where they steepen the solution rises or falls faster; equal slopes across every horizontal row suggest dependence only on y; equal slopes down every vertical column suggest dependence only on x.



Slope field of $\dfrac{dM}{dt} = \frac{1}{4}(40 - M)$*: segments are flat along* $M = 40$*, rise below it and fall above it. The solution through* $(0, 5)$ *climbs quickly, then levels off towards* 40*.*

Reading the field above. The right-hand side depends only on M, so every horizontal row of segments is identical. Where $M = 40$ the slope is 0 at every t, so the

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horizontal line $M = 40$ is itself a solution: an **equilibrium solution** 平衡解. Below it $\dfrac{dM}{dt} > 0$ (solutions rise towards 40); above it $\dfrac{dM}{dt} < 0$ (solutions fall towards 40). A solution curve never crosses the equilibrium line; it only approaches it.

Concavity from the equation. Differentiate the differential equation itself, then substitute $\dfrac{dM}{dt}$ back in:

$$\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\cdot\frac{1}{4}(40 - M) = -\frac{1}{16}(40 - M).$$

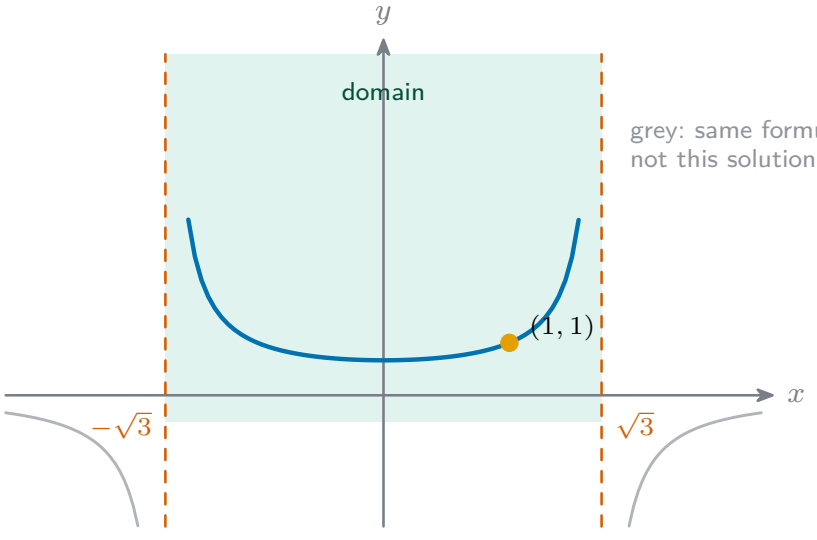
For $5 \le M < 40$ this is negative, so the solution through $(0, 5)$ is **concave down** 下凹, as the picture shows.

Finding General Solutions Using Separation of Variables

Many exam differential equations are solved by **separation of variables** 分离变量法. If $\dfrac{dy}{dx}$ factors into a function of x times a function of y, move all y's to one side and all x's to the other, then integrate both sides where the divided factor is nonzero:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

Add the **constant of integration** 积分常数 once (on the x-side). This gives the **general solution** 通解. Also check **constant solutions** 常数解 with $h(y) = 0$ separately; division by $h(y)$ can lose them.



The solution lives on the interval containing the initial x.

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Worked example. Solve $\dfrac{dy}{dx} = xy$ with the initial condition $y(0) = 2$. Separate and integrate:

$$\int \frac{dy}{y} = \int x\,dx \;\Rightarrow\; \ln|y| = \frac{x^2}{2} + C \;\Rightarrow\; y = Ae^{x^2/2}.$$

Applying $y(0) = 2$ gives $A = 2$, so the particular solution is $y = 2e^{x^2/2}$.

Common mistakes. Adding the constant at the end: $y = e^{x^2/2} + C$ is wrong, because the constant enters when you integrate and becomes the multiplier $A = e^C$. Trying to separate an equation such as $\dfrac{dy}{dx} = x + y$: a sum cannot be written as $g(x)\,h(y)$.

Worked example: a power of y. Find the particular solution of $\dfrac{dy}{dx} = xy^2$ with $y(1) = 1$, and state its domain.

$$\int y^{-2}\,dy = \int x\,dx \;\Rightarrow\; -\frac{1}{y} = \frac{x^2}{2} + C.$$

At $(1, 1)$: $-1 = \frac{1}{2} + C$, so $C = -\frac{3}{2}$. Then $-\dfrac{1}{y} = \dfrac{x^2 - 3}{2}$, so

$$y = \frac{2}{3 - x^2}.$$

The formula breaks at $x = \pm\sqrt{3}$. The solution must live on one interval that contains the initial $x = 1$, so its domain is $-\sqrt{3} < x < \sqrt{3}$.

Finding Particular Solutions Using Initial Conditions and Separation of Variables

An **initial condition** 初始条件 – a known point (x_0, y_0) – pins down one curve from the family. Substitute it to solve for the constant C; the result is the **particular solution** 特解. **Uniqueness** 唯一性 requires conditions: for $y' = F(x, y)$, **continuity** 连续性 of F and its **partial derivative** 偏导数 with respect to y near the initial point is a sufficient local condition. Watch for **domain restrictions** 定义域限制: keep the branch that contains the initial point (for example, the correct sign of a square root).

Worked example. Solve $\dfrac{dy}{dx} = \dfrac{x}{y}$ with $y(0) = -3$.

$$\int y\,dy = \int x\,dx \;\Rightarrow\; \frac{y^2}{2} = \frac{x^2}{2} + C.$$

Substitute $x = 0$, $y = -3$ **now**, before solving for y: $\frac{9}{2} = 0 + C$, so $C = \frac{9}{2}$ and

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$y^2 = x^2 + 9$. Taking the square root gives $y = \pm\sqrt{x^2+9}$; the initial value $y(0) = -3$ is negative, so the particular solution is

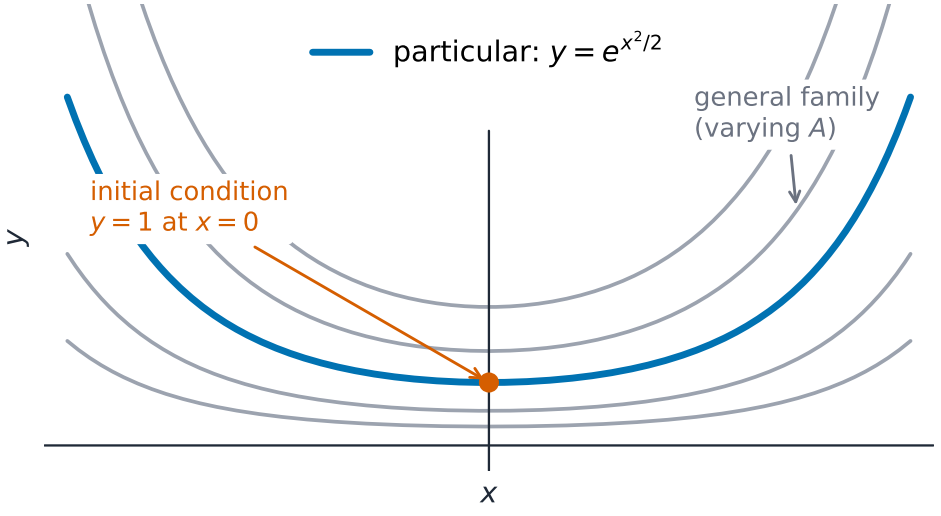
$$y = -\sqrt{x^2+9}.$$

Writing $y = +\sqrt{x^2+9}$ is the classic error: it fails the initial condition.

When there is no y on the right. If $\dfrac{dy}{dx} = f(x)$ and $y(a) = y_0$, the Fundamental Theorem of Calculus gives the particular solution directly:

$$y(x) = y_0 + \int_a^x f(t)\,dt.$$

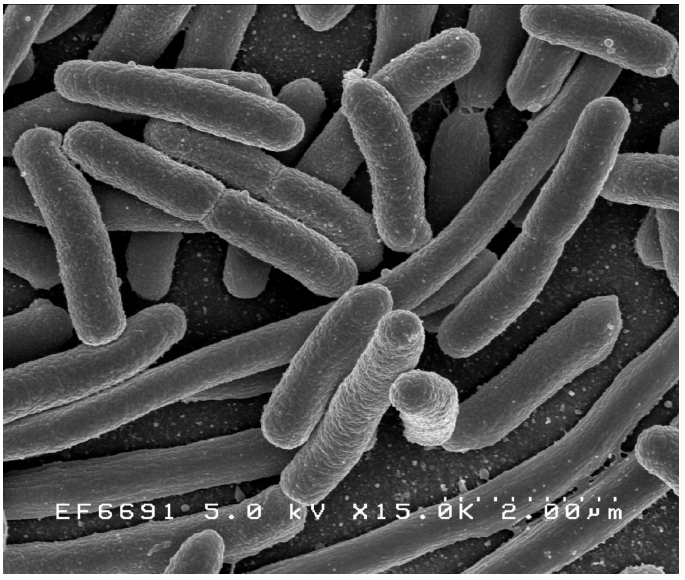
This works even when f has no antiderivative you can write down. For example, if $\dfrac{dy}{dx} = \sin(x^2)$ and $y(0) = 4$, then $y(2) = 4 + \displaystyle\int_0^2 \sin(t^2)\,dt \approx 4 + 0.805 = 4.805$ (calculator).



The constant gives a family of curves; an initial condition picks out one

Exponential Models with Differential Equations

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E. coli under a microscope: differential equations model exponential growth of populations

The most important model is **exponential growth and decay** 指数增长与衰减. The equation $\dfrac{dy}{dt} = ky$ (rate proportional to size) has the solution:

$$y = y_0\, e^{kt},$$

where y_0 is the value at $t = 0$. Here $k > 0$ gives growth and $k < 0$ gives decay. You can derive this by separation of variables ($\int \frac{dy}{y} = \int k\, dt$), and it models processes like population growth and **radioactive decay** 放射性衰变. In **Newton's law of cooling** 牛顿冷却定律 with fixed ambient temperature, apply this exponential model to the temperature difference from the surroundings. A follow-up part may ask for $\dfrac{d^2y}{dt^2}$: differentiate $\dfrac{dy}{dt} = ky$ again (using $\dfrac{dy}{dt} = ky$) to express it in terms of y.

Worked example. A sample decays by $\dfrac{dy}{dt} = -0.10\, y$ (in years) from $y_0 = 50$ g. The solution is $y = 50e^{-0.10t}$, so after 10 years $y = 50e^{-1} = 18.4$ g. Its **half-life** 半衰期 solves $25 = 50e^{-0.10t}$, i.e. $e^{-0.10t} = \frac{1}{2}$, giving $t = \dfrac{\ln 2}{0.10} = 6.9$ years – independent of the starting amount.

Worked example: Newton's law of cooling. Coffee at 90°C is left in a room at 20°C. Its temperature T obeys $\dfrac{dT}{dt} = -k\,(T - 20)$, with t in minutes. After 5 minutes it is 70°C. When does it reach 40°C?

The temperature difference $D = T - 20$ decays exponentially: $D = 70e^{-kt}$ (the start value is $90 - 20 = 70$), so

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$$T = 20 + 70e^{-kt}.$$

Use $T(5) = 70$: $50 = 70e^{-5k}$, so $k = \dfrac{\ln(7/5)}{5} \approx 0.0673$ per minute. For $T = 40$:

$$20 = 70e^{-kt} \;\Rightarrow\; t = \frac{\ln 3.5}{k} \approx \frac{1.2528}{0.06729} \approx 18.6 \text{ minutes}.$$

Common error: writing $T = 90e^{-kt}$, which would cool the coffee towards 0°C instead of towards room temperature.



A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

Worked example: limited growth. Medicine enters the blood so that the amount M, in milligrams, satisfies $\dfrac{dM}{dt} = \frac{1}{4}(40 - M)$, with t in hours and $M(0) = 5$.

(a) *Solve.* Separate: $\displaystyle\int \frac{dM}{40 - M} = \int \frac{1}{4}\,dt$, so $-\ln|40 - M| = \dfrac{t}{4} + C$. At $t = 0$: $-\ln 35 = C$. Since $M < 40$, $|40 - M| = 40 - M$, and

$$40 - M = 35e^{-t/4} \;\Rightarrow\; M = 40 - 35e^{-t/4}.$$

(b) *Long-run behaviour.* As $t \to \infty$, $e^{-t/4} \to 0$, so $\lim\limits_{t\to\infty} M(t) = 40$. In context: over a long time the amount of medicine in the blood approaches 40 milligrams. Always give the units and the context.

(c) *Tangent-line estimate.* $M'(0) = \frac{1}{4}(40 - 5) = 8.75$, so the **linear approximation** 线性近似 at $t = 0$ gives $M(2) \approx 5 + 8.75 \cdot 2 = 22.5$ mg. Is this too big or too small? From the slope field section, $\dfrac{d^2M}{dt^2} = -\frac{1}{16}(40 - M) < 0$, so the graph is concave

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down and lies **below** its tangent line: the estimate is an **overestimate** 高估. (Check: $M(2) = 40 - 35e^{-0.5} \approx 18.8$ mg.)

The pattern $\dfrac{dy}{dt} = k(L - y)$ always gives $y = L - (L - y_0)e^{-kt}$: the gap $L - y$ decays exponentially, exactly like Newton's law of cooling.

Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope – you must integrate to recover the function.

Practice.

1. Write a differential equation: the rate at which a rumour spreads in a school of 1000 students is proportional to the product of the number N who have heard it and the number who have not.
2. Verify that $y = 3e^{2x} - 1$ is a solution of $\dfrac{dy}{dx} = 2y + 2$.
3. For $\dfrac{dy}{dx} = y - x$, find the slope of the segment at $(2, 5)$, and describe every point where the segments are horizontal.
4. Find the particular solution of $\dfrac{dy}{dx} = \dfrac{x^2}{y}$ with $y(0) = -2$.
5. The amount A of salt, in grams, in a tank satisfies $\dfrac{dA}{dt} = \dfrac{12 - A}{3}$ with $A(0) = 0$, t in minutes. Find $A(t)$ and $\lim\limits_{t \to \infty} A(t)$, and interpret the limit.
6. $\dfrac{dy}{dt} = 0.5(y - 10)$ with $y(0) = 4$. Find $\dfrac{d^2y}{dt^2}$ in terms of y, use the tangent line at $t = 0$ to estimate $y(1)$, and decide whether the estimate is too big or too small.

Answers.

1. $\dfrac{dN}{dt} = kN(1000 - N)$.
2. $\dfrac{dy}{dx} = 6e^{2x}$ and $2y + 2 = 6e^{2x} - 2 + 2 = 6e^{2x}$; the sides agree for all x.
3. Slope $5 - 2 = 3$. Horizontal where $y - x = 0$: every point on the line $y = x$.
4. $\dfrac{y^2}{2} = \dfrac{x^3}{3} + C$; at $(0, -2)$, $C = 2$, so $y^2 = \dfrac{2x^3}{3} + 4$. Since $y(0) < 0$, $y = -\sqrt{\dfrac{2x^3}{3} + 4}$, for $x > -\sqrt[3]{6}$.

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5. $A = 12 - 12e^{-t/3}$; the limit is 12: over a long time the tank holds close to 12 grams of salt.
6. $\dfrac{d^2y}{dt^2} = 0.5\dfrac{dy}{dt} = 0.25(y-10)$. $y'(0) = -3$, so $y(1) \approx 4-3 = 1$. For $y < 10$, $\dfrac{d^2y}{dt^2} < 0$: concave down, so the estimate is too big (exact: $10 - 6e^{0.5} \approx 0.11$).

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Applications of Integration

AP Calculus AB

Average Value of a Function

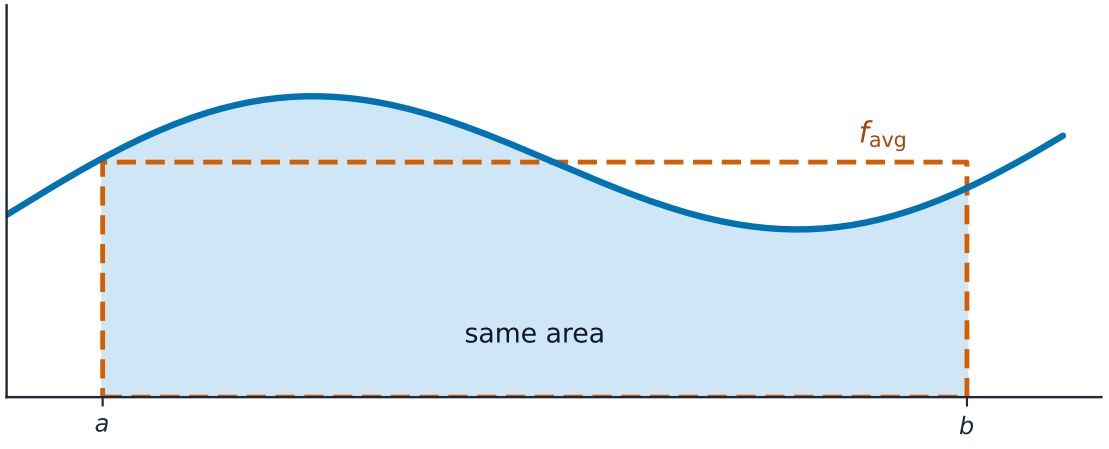
The **average value** 平均值 of a **continuous function** 连续函数 f over $[a, b]$ is the **integral** 积分 divided by the **interval** 区间 length:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It is the constant height of a **rectangle** 矩形 on $[a, b]$ with the same area as under f. Do not confuse it with the **average rate of change** 平均变化率 (which divides *change in* f by the interval), or with a **Riemann sum** 黎曼和 average of a few values – the exam distinguishes these carefully. Report units in context (e.g. the average rate "in vehicles per hour").

Worked example. The average value of $f(x) = x^2$ on $[0, 3]$ is

$$f_{\text{avg}} = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}(9) = 3.$$



The rectangle of height f_avg has the same area as the region under f

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Position, Velocity, and Acceleration Using Integrals

Integration reverses the motion links of Unit 4. For a **particle** 质点 in straight-line motion over $[t_1, t_2]$:



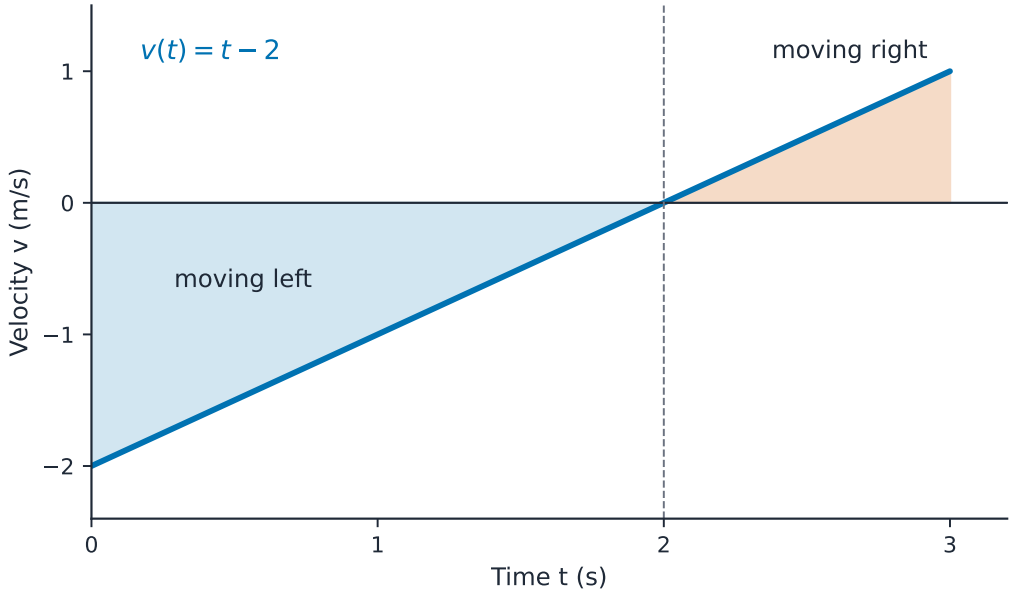
A speedometer shows speed at each instant; integrating velocity over time gives displacement

- **Displacement** 位移 (net change in position) $= \displaystyle\int_{t_1}^{t_2} v(t)\,dt.$
- **Total distance travelled** 总路程 $= \displaystyle\int_{t_1}^{t_2} |v(t)|\,dt$ – integrate **speed** 速率, so direction changes add up instead of cancelling.
- **Position** at a later time $= x(t_1) + \displaystyle\int_{t_1}^{t} v(s)\,ds$; **velocity** 速度 from **acceleration** 加速度 $v(t) = v(t_1) + \int_{t_1}^{t} a(u)\,du$.

The displacement-versus-distance distinction (with the **absolute value** 绝对值) is a classic graded point.

■ **Read the sign before adding motion**

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Teacher-created model: velocity is negative before two seconds and positive afterwards; the particle changes direction.

For $v(t) = t - 2$ m/s on $[0, 3]$, solve $v(t) = 0$ and test either side. The particle moves left on $[0, 2)$ and right on $(2, 3]$.

Displacement is signed:

$$\Delta x = \int_0^3 v(t)\,dt = \left[\frac{t^2}{2} - 2t\right]_0^3 = -1.5 \text{ m}.$$

Total distance counts both journeys positively. Split at the **sign change** 符号变化:

$$D = \int_0^3 |v(t)|\,dt = -\int_0^2 (t-2)\,dt + \int_2^3 (t-2)\,dt.$$

$$D = |-2| + \frac{1}{2} = 2.5 \text{ m}.$$

The particle travels two metres left and half a metre right. Taking only $|\int_0^3 v(t)\,dt| = 1.5$ m loses the return journey.

Position needs a starting position. If $x(0) = 5$ m, then

$$x(3) = x(0) + \int_0^3 v(t)\,dt = 5 - 1.5 = 3.5 \text{ m}.$$

The final position is right of the **origin** 原点, even though displacement is negative.

A zero of velocity is a candidate for a turn. Check signs: $v(t) = (t-1)^2$ is positive on both sides of $t = 1$, so the particle stops momentarily without changing direction.

■ Recover velocity with an initial condition

For $a(t) = 4$ m/s^2 and $v(0) = 1$ m/s:

$$v(t) = v(0) + \int_0^t a(u)\,du = 1 + \int_0^t 4\,du = 1 + 4t.$$

Here t is in seconds and $v(t)$ is in metres per second. Equivalently, integrate to $4t + C$ and use $v(0) = 1$ to find $C = 1$. Each integration needs its own **initial condition** 初始条件.

■ Original exam applications

2025 AB Q5D (calculator not allowed). The original gives $v_J(t) = 2t(t^2-1)^3$ and $x_J(0) = 7$, without physical units. Recover position, using $u = t^2 - 1$ and $du = 2t\,dt$:

$$x_J(2) = x_J(0) + \int_0^2 v_J(t)\,dt = 7 + \int_{-1}^3 u^3\,du.$$

$$x_J(2) = 7 + \left[\frac{u^4}{4}\right]_{-1}^3 = 7 + \frac{81-1}{4} = 27.$$

The change in position is 20, while final position is 27. Keep the initial position outside the integral.

2014 AB Q4(c). At times $2, 5, 8, 12$ min, train velocities are $100, 40, -120, -150$ m/min. Starting position is $s_A(2) = 300$ m east of the origin. Write the exact position expression, then estimate the signed change by the three specified **trapezoids** 梯形:

$$s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt.$$

$$\Delta s \approx 3\frac{100+40}{2} + 3\frac{40-120}{2} + 4\frac{-120-150}{2} = -450 \text{ m}.$$

$$s_A(12) \approx 300 - 450 = -150 \text{ m}.$$

This means about 150 m west of the origin. Preserve the unequal interval widths and signs. Sampled velocities do not locate every zero, so absolute trapezoids do not give exact total distance.

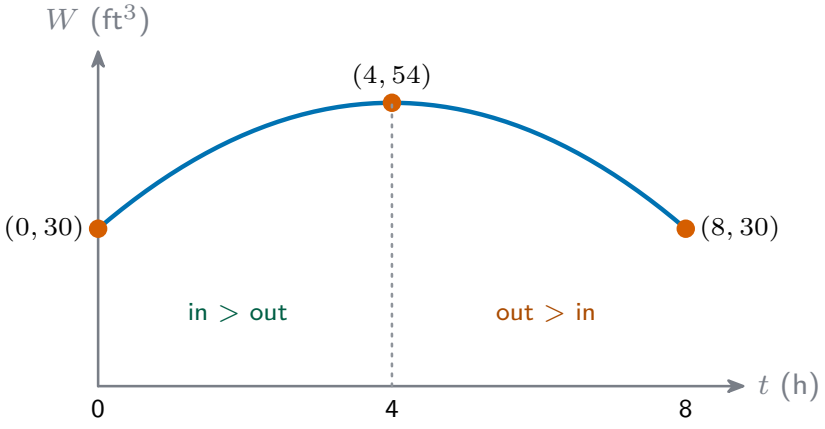
Accumulation Functions and Definite Integrals in Applied Contexts

A function defined as an integral accumulates a rate of change. The **net change** 净变化 theorem is the everyday tool: the **definite integral** 定积分 of a rate over an interval gives the net change of the quantity:

$$(\text{final}) = (\text{initial}) + \int_a^b (\text{rate})\,dt.$$

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So "how much water is in the tank at $t = 5$" = starting amount + $\int_0^5$(inflow − outflow) dt. Watch signs (in minus out) and units. Many multi-part FRQs are built entirely on this idea – often paired with "write, but do not evaluate, an integral expression that gives the total...".



Initial amount plus the integral of rate in minus rate out.

Worked example (rate in, rate out). Water flows into a pipe at 12 cubic feet per hour and drains out at $D(t) = 3t$ cubic feet per hour, for $0 \le t \le 8$. At $t = 0$ the pipe holds 30 cubic feet.

(a) *Amount.* $W(t) = 30 + \displaystyle\int_0^t (12 - 3s)\,ds = 30 + 12t - 1.5t^2$.

(b) *Increasing or decreasing at $t = 3$?* Compare the rates, not the amounts: $W'(3) = 12 - 3(3) = 3 > 0$, so the amount is increasing, because water enters faster than it leaves.

(c) *Minimum amount.* $W'(t) = 0$ at $t = 4$. Use the **candidates test** – check the critical point and both endpoints:

t	0	4	8
$W(t)$	30	54	30

The minimum is 30 cubic feet, at $t = 0$ and $t = 8$; $t = 4$ is the maximum. A sign chart of W' alone can find the maximum, but only the table settles which endpoint is lowest.

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Finding the Area Between Curves (Functions of x)

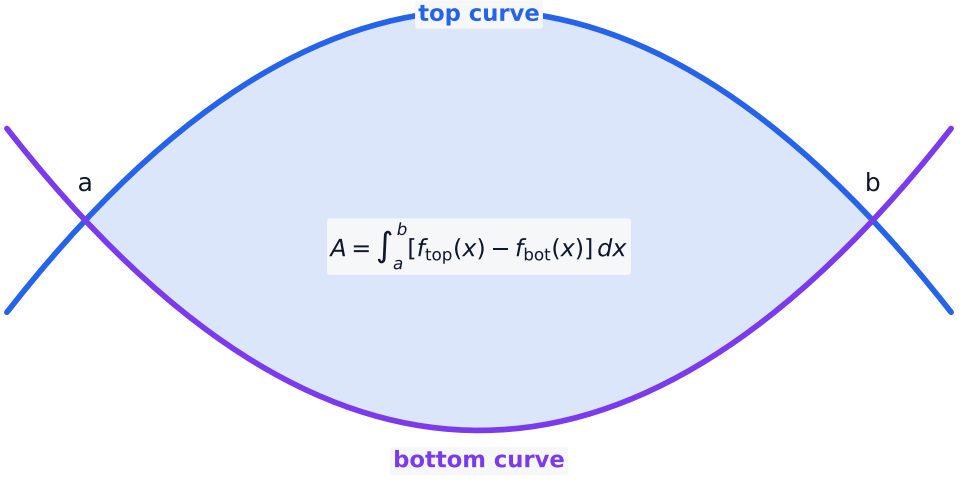
The area between two curves $y = f(x)$ (top) and $y = g(x)$ (bottom) on $[a, b]$ is:

$$A = \int_a^b \big[f(x) - g(x)\big]\,dx = \int_a^b (\text{top} - \text{bottom})\,dx.$$

Find the **intersection points** 交点 first to set the **limits of integration** 积分限, and always subtract bottom from top.

Worked example. Find the area between $y = x$ and $y = x^2$. They cross at $x = 0$ and $x = 1$, and on $(0, 1)$ the line $y = x$ is on top, so

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$



The shaded region runs from one intersection a to the next b; its area is $\int_a^b (top - bottom)\,dx$.

Finding the Area Between Curves (Functions of y)

Some regions are easier described with **horizontal slices** 水平切片 – integrate with respect to y:

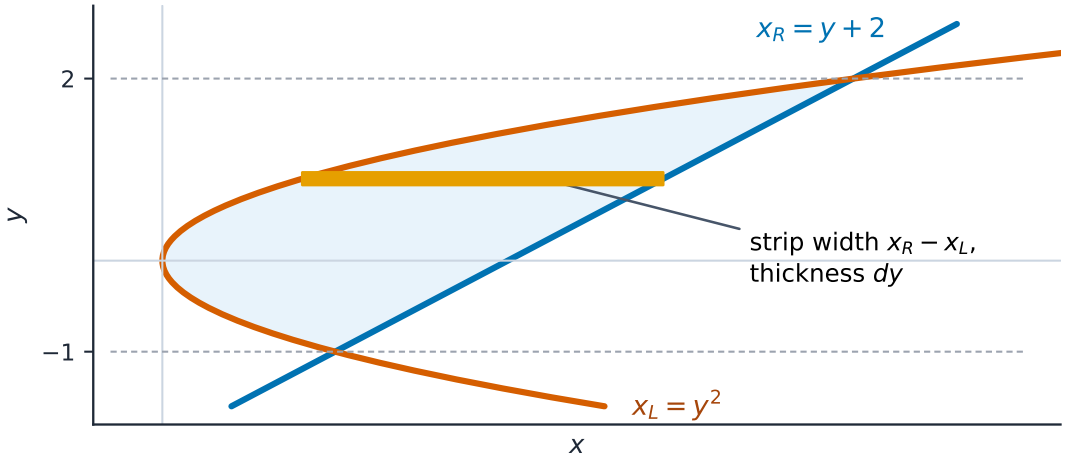
$$A = \int_c^d \big[x_{\text{right}}(y) - x_{\text{left}}(y)\big]\,dy = \int_c^d (\text{right} - \text{left})\,dy.$$

Choose x- or y-slices to keep a single top/bottom (or right/left) pair over the whole region.

Worked example. Find the same area between $y = x$ and $y = x^2$ with horizontal slices. Solve each curve for x: the line is $x = y$ and the **parabola** 抛物线 is $x = \sqrt{y}$. For $0 < y < 1$, $\sqrt{y} > y$ (try $y = 0.25$: $0.5 > 0.25$), so the parabola is on the right:

$$A = \int_0^1 \left(\sqrt{y} - y\right) dy = \left[\frac{2}{3}y^{3/2} - \frac{y^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

Both methods give $\frac{1}{6}$, as they must.



$$A = \int_{-1}^{2} [(y + 2) - y^2]\, dy$$

With x as a function of y, use horizontal strips: right curve minus left curve

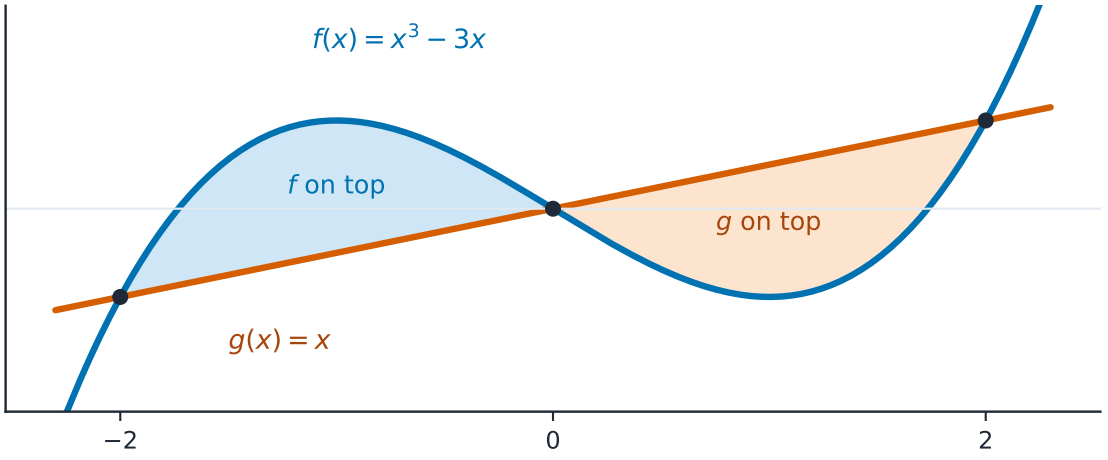
Area Between Curves With More Than Two Intersections

If the curves cross more than twice, one function is on top for part of the region and the other on top elsewhere. Split into a **sum of integrals** at each crossing, or integrate the **absolute value of the difference**:

$$A = \int_a^b \left|f(x) - g(x)\right| dx.$$

Determine which curve is higher on each subinterval before writing the integrals.

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$$A = \int_{-2}^{0} (f - g)\,dx + \int_{0}^{2} (g - f)\,dx = \int_{-2}^{2} |f - g|\,dx$$

When curves cross more than twice, split at each crossing so every piece is top minus bottom

Worked example. Find the area between $y = x^3$ and $y = x$. They meet where $x^3 = x$, at $x = -1, 0, 1$. Test a point in each piece: at $x = -\frac{1}{2}$, $x^3 = -\frac{1}{8} > -\frac{1}{2}$, so the cubic is on top; at $x = \frac{1}{2}$ the line is on top.

$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Common mistake. Writing $\displaystyle\int_{-1}^{1} (x - x^3)\,dx = 0$: the two pieces cancel. On the calculator, $\displaystyle\int_{-1}^{1} \left|x - x^3\right|\,dx$ gives $\frac{1}{2}$ in one step.

Volumes with Cross Sections: Squares and Rectangles

If a solid has a known **cross section** 横截面 **perpendicular** 垂直 to an axis, its **volume** 体积 is the integral of the cross-sectional **area**:

$$V = \int_{a}^{b} A(x)\,dx.$$

For **square** 正方形 cross sections with side equal to the region's height $s(x) = f(x) - g(x)$, use $A(x) = [s(x)]^2$; for a **rectangle** of height $k \cdot s(x)$, use $A(x) = k\,[s(x)]^2$. The exam phrase "the base of a solid... cross sections are squares/rectangles" signals exactly this.

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A potter shaping a vessel: solids of revolution are volumes formed by rotating a plane region

Worked example. The base of a solid is the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$, and cross sections perpendicular to the x-axis are squares (the solid in the figure below). Each square's side is $s(x) = \sqrt{x}$, so $A(x) = x$ and

$$V = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = 8.$$

With semicircles on the same base instead, $A(x) = \frac{\pi}{8}\big(\sqrt{x}\big)^2$, so $V = \frac{\pi}{8}\cdot 8 = \pi$: only the area formula changes.

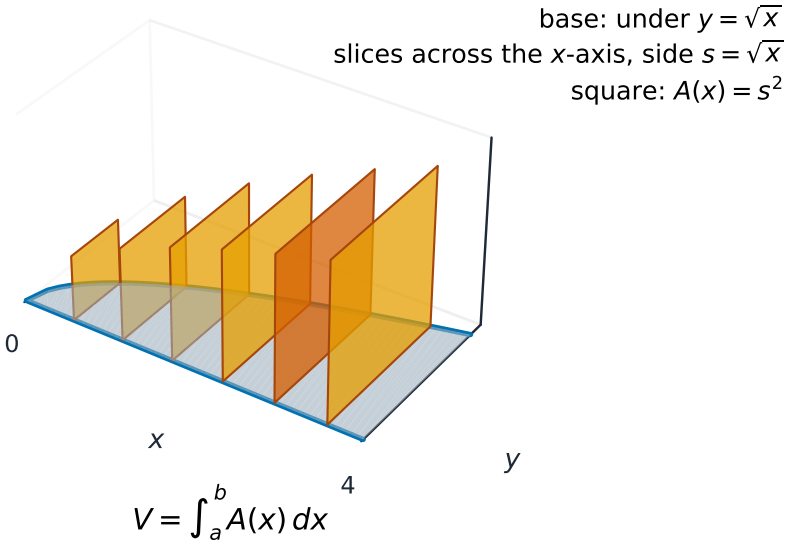
Worked example. The base of a solid is the region between $y = x$ and $y = x^2$. Cross sections perpendicular to the x-axis are squares. Find the volume.

The side of each square is the slice length $s(x) = x - x^2$, so $A(x) = (x - x^2)^2 = x^2 - 2x^3 + x^4$ and

$$V = \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{10 - 15 + 6}{30} = \frac{1}{30}.$$

Expand the square before integrating; $\int (x - x^2)^2\,dx$ is **not** $\left(\int (x - x^2)\,dx\right)^2$.

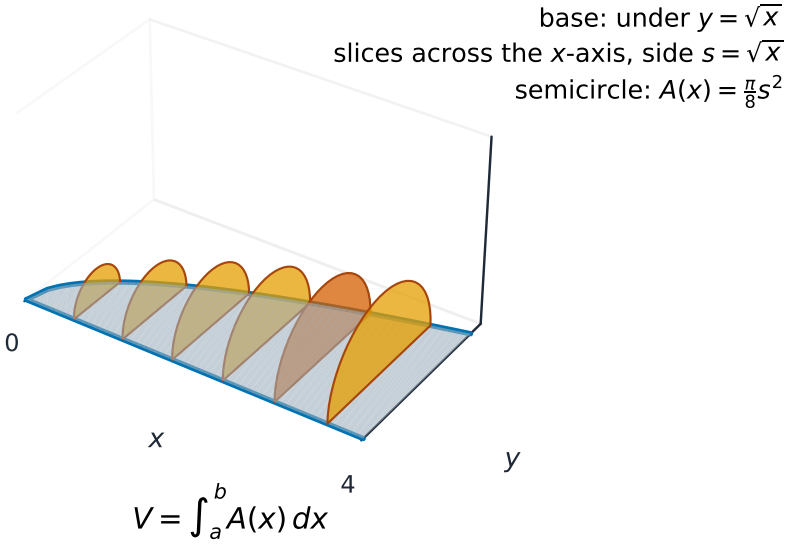
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Square cross sections on a base region: add the slice areas $A(x)$ along the x-axis

Volumes with Cross Sections: Triangles and Semicircles

Same method, different area formula: for an equilateral **triangle** 三角形 of side s, $A = \frac{\sqrt{3}}{4}s^2$; for a **semicircle** 半圆 of **diameter** 直径 s, $A = \frac{\pi}{8}s^2$. Set the shape's key length equal to the region's slice length $s(x)$, then integrate $A(x)$.



Semicircular cross sections on the same base: only the area formula $A(x)$ changes

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Worked example. The base is the region between $y = x$ and $y = x^2$ for $0 \le x \le 1$. Cross sections perpendicular to the x-axis have side (or diameter) $s = x - x^2$, and the squares section found $\displaystyle\int_0^1 (x - x^2)^2\,dx = \frac{1}{30}$.

- Equilateral triangles: $V = \dfrac{\sqrt{3}}{4}\displaystyle\int_0^1 (x - x^2)^2\,dx = \frac{\sqrt{3}}{4}\cdot\frac{1}{30} = \frac{\sqrt{3}}{120}$.
- Semicircles: $V = \dfrac{\pi}{8}\cdot\dfrac{1}{30} = \dfrac{\pi}{240}$.

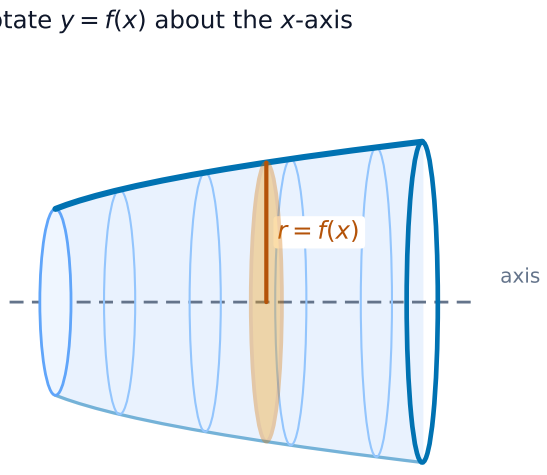
Only the constant in front changes: square 1, equilateral triangle $\frac{\sqrt{3}}{4}$, semicircle $\frac{\pi}{8}$. The semicircle area is $\frac{1}{2}\pi r^2$ with radius $r = \frac{s}{2}$: $\frac{1}{2}\pi\frac{s^2}{4} = \frac{\pi}{8}s^2$. Using s as the radius is the usual slip.

Volume with the Disc Method (About the x- or y-Axis)

Revolving a region around an axis makes a **solid of revolution** 旋转体. If the region touches the axis, each slice is a **disc** 圆盘 of **radius** 半径 r = the function value:

$$V = \pi\int_a^b [r(x)]^2\,dx.$$

Around the x-axis, $r = f(x)$ and integrate in x; around the y-axis, express x as a function of y and integrate in y.

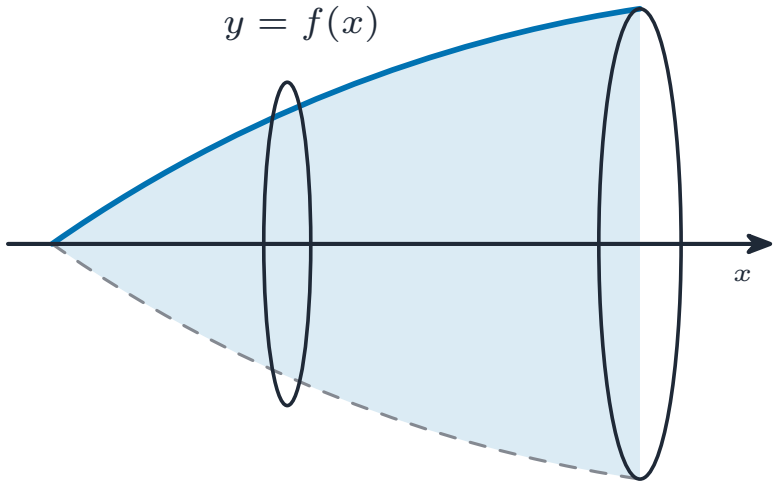


The disc method: rotating y=f(x) about the axis sweeps out disks of radius f(x)

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Worked example. Revolve the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ about the x-axis. Each disc has radius $r = \sqrt{x}$, so

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \pi \left[\frac{x^2}{2} \right]_0^4 = 8\pi.$$



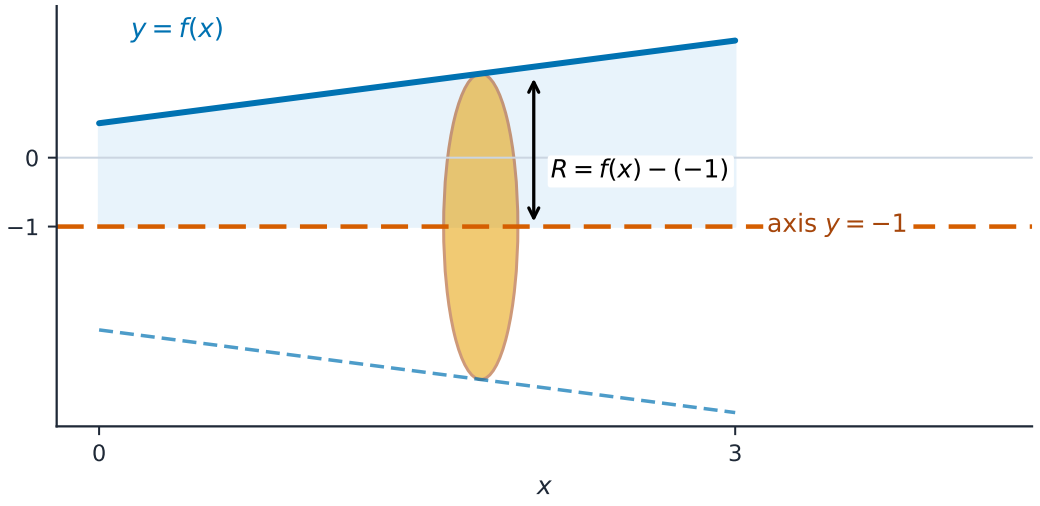
rotate the region around the x-axis to sweep a solid

Rotating a region about an axis sweeps out a solid of revolution

Volume with the Disc Method (About Other Axes)

Around any horizontal or vertical line $y = k$ or $x = k$, the radius is the **distance** from the curve to that line, e.g. $r(x) = |f(x) - k|$. Set up the radius carefully, then use $V = \pi \int r^2$.

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$$V = \pi \int_0^3 [f(x) + 1]^2 dx$$

Rotating about $y = -1$: measure the radius from the axis, $R = f(x) + 1$

Worked example. The region between $y = \sqrt{x}$ and the line $y = 2$, for $0 \le x \le 4$, is revolved about $y = 2$. Each disc has radius $R = 2 - \sqrt{x}$ (line minus curve, since the line is above):

$$V = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx = \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \pi \left(16 - \frac{64}{3} + 8\right) = \frac{8\pi}{3}.$$

For a **write, but do not evaluate** question, the integral with correct limits and radius is the whole answer – do not waste time evaluating it.

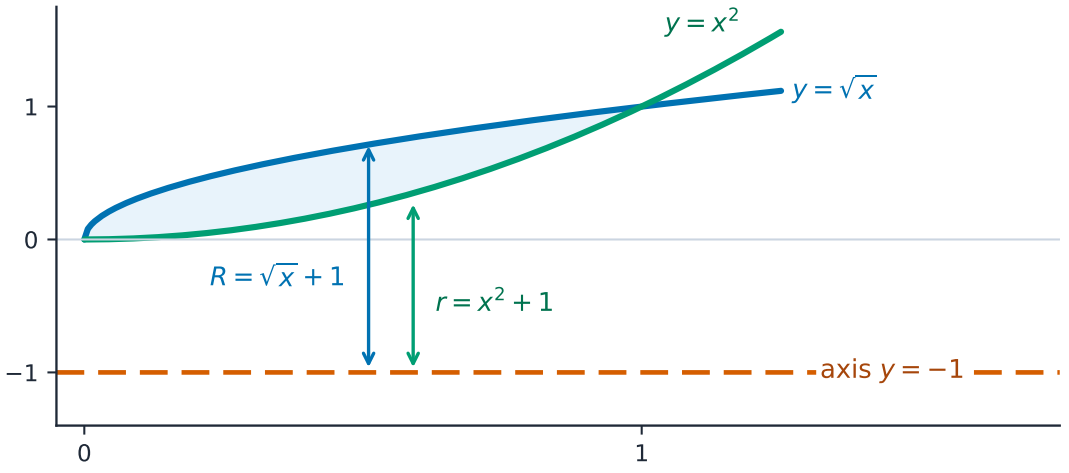
Volume with the Washer Method

When the region does **not** touch the axis, each slice is a **washer** 垫圈 (a ring) with an outer radius R and inner radius r:

$$V = \pi \int_a^b \left([R]^2 - [r]^2\right) dx.$$

R is the distance from the axis to the **farther** boundary and r to the **nearer** one. Around a line other than an axis, both radii are measured as distances to that line – a very common exam variation (e.g. revolving about $y = -2$).

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$V = \pi \int_0^1 [R^2 - r^2]\,dx$, with each radius = curve − axis

Washers about $y = -1$: outer and inner radii are both measured from the axis

Worked example. Revolve the region between $y = \sqrt{x}$ and $y = x^2$ $(0 \le x \le 1)$ (the region in the figure) about (a) the x-axis and (b) the line $y = -1$.

(a) On $(0, 1)$, $\sqrt{x} > x^2$, so $y = \sqrt{x}$ is farther from the x-axis: $R = \sqrt{x}$ and $r = x^2$.

$$V = \pi \int_0^1 \left(x - x^4\right) dx = \pi \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}.$$

(b) Each radius is now a distance to $y = -1$, so add 1 to each height: $R = \sqrt{x} + 1$ and $r = x^2 + 1$. Expand both squares:

$$(\sqrt{x} + 1)^2 - (x^2 + 1)^2 = \left(x + 2\sqrt{x} + 1\right) - \left(x^4 + 2x^2 + 1\right) = x + 2\sqrt{x} - x^4 - 2x^2.$$

$$V = \pi \int_0^1 \left(x + 2\sqrt{x} - x^4 - 2x^2\right) dx = \pi \left(\frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3}\right) = \frac{29\pi}{30}.$$

Common mistakes. Writing $\pi \int (R - r)^2\,dx$ instead of $\pi \int (R^2 - r^2)\,dx$; and, about $y = -1$, using $R = \sqrt{x} - 1$ (subtracting) instead of $R = \sqrt{x} - (-1) = \sqrt{x} + 1$.

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Exam tips

- Area between curves is $\int(\text{top}-\text{bottom})\,dx$ – find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2-r^2)$).
- The **average value** of f on $[a,b]$ is $\frac{1}{b-a}\int_a^b f\,dx$.
- Accumulated change is $\int$ of a rate: total = initial value $+\int_a^b(\text{rate})\,dt$.
- Integration means "adding up infinitely many tiny pieces" – set up the integrand as one thin slice.

Practice.

1. Find the average value of $f(x)=\sin x$ on $[0,\pi]$.
2. $v(t)=2t-4$ m/s for $0\le t\le 5$. Find the displacement and the total distance.
3. Find the area enclosed by $y=x^2$ and $y=4$.
4. Find the volume when the region under $y=x^2$, $0\le x\le 2$, is revolved about the x-axis.
5. The region between $y=x$ and $y=x^2$ is revolved about the line $y=-1$. Write, but do not evaluate, an integral for the volume.

Answers.

1. $\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}$.
2. Displacement $\displaystyle\int_0^5(2t-4)\,dt=25-20=5$ m. $v<0$ on $(0,2)$: distance $=4+9=13$ m.
3. They meet at $x=\pm 2$: $\displaystyle\int_{-2}^{2}(4-x^2)\,dx=16-\frac{16}{3}=\frac{32}{3}$.
4. $\pi\displaystyle\int_0^2 x^4\,dx=\frac{32\pi}{5}$.
5. $\pi\displaystyle\int_0^1\left[(x+1)^2-(x^2+1)^2\right]dx$.

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