Applications of Integration

AP Calculus AB

Average Value of a Function

The **average value** 平均值 of a continuous function f over $[a, b]$ is the integral divided by the interval length:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It is the constant height of a rectangle on $[a, b]$ with the same area as under f. Do not confuse it with the **average rate of change** (which divides *change in* f by the interval), or with a Riemann-sum average of a few values – the exam distinguishes these carefully. Report units in context (e.g. the average rate "in vehicles per hour").

Worked example. The average value of $f(x) = x^2$ on $[0, 3]$ is

$$f_{\text{avg}} = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}(9) = 3.$$

Position, Velocity, and Acceleration Using Integrals

Integration reverses the motion links of Unit 4. For a particle in straight-line motion over $[t_1, t_2]$:

- **Displacement** 位移 (net change in position) $= \displaystyle\int_{t_1}^{t_2} v(t)\,dt$.
- **Total distance travelled** 总路程 $= \displaystyle\int_{t_1}^{t_2} |v(t)|\,dt$ – integrate **speed**, so direction changes add up instead of cancelling.
- **Position** at a later time $= x(t_1) + \displaystyle\int_{t_1}^{t} v(s)\,ds$; **velocity** from acceleration $= v(t_1) + \int a$.

The displacement-versus-distance distinction (with the absolute value) is a classic graded point.

Accumulation Functions and Definite Integrals in Applied Contexts

A function defined as an integral accumulates a rate of change. The **net change** 净变化 theorem is the everyday tool: the definite integral of a rate over an interval gives the net change of the quantity:

$$(\text{final}) = (\text{initial}) + \int_a^b (\text{rate})\,dt.$$

So "how much water is in the tank at $t = 5$" = starting amount + $\int_0^5(\text{inflow} - \text{outflow})\,dt$. Watch signs (in minus out) and units. Many multi-part FRQs are built entirely on this idea – often paired with "write, but do not evaluate, an integral expression that gives the total...".

Finding the Area Between Curves (Functions of x)

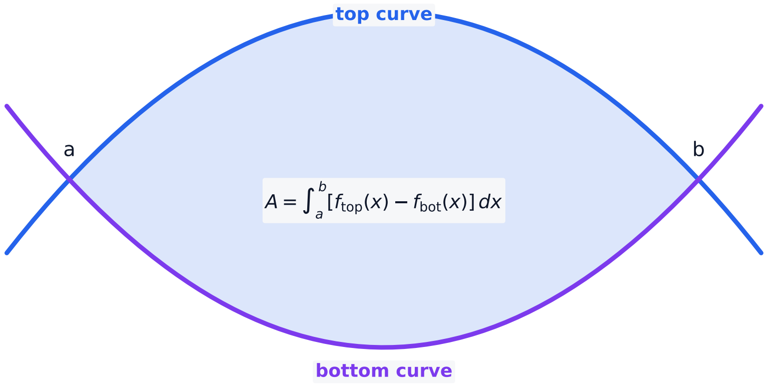
The area between two curves $y = f(x)$ (top) and $y = g(x)$ (bottom) on $[a, b]$ is:

$$A = \int_a^b \big[f(x) - g(x)\big]\,dx = \int_a^b (\text{top} - \text{bottom})\,dx.$$

Find the intersection points first to set the limits, and always subtract bottom from top.

Worked example. Find the area between $y = x$ and $y = x^2$. They cross at $x = 0$ and $x = 1$, and on $(0, 1)$ the line $y = x$ is on top, so

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$



The shaded region runs from one intersection a to the next b; its area is $\int_a^b (top - bottom)\,dx$.

Finding the Area Between Curves (Functions of y)

Some regions are easier described with **horizontal** slices – integrate with respect to y:

$$A = \int_c^d \left[x_{\text{right}}(y) - x_{\text{left}}(y)\right] dy = \int_c^d (\text{right} - \text{left})\,dy.$$

Choose x- or y-slices to keep a single top/bottom (or right/left) pair over the whole region.

Area Between Curves With More Than Two Intersections

If the curves cross more than twice, one function is on top for part of the region and the other on top elsewhere. Split into a **sum of integrals** at each crossing, or integrate the **absolute value of the difference**:

$$A = \int_a^b \left|f(x) - g(x)\right| dx.$$

Determine which curve is higher on each subinterval before writing the integrals.

Volumes with Cross Sections: Squares and Rectangles

A potter shaping a vessel: solids of revolution are volumes formed by rotating a plane region

If a solid has a known **cross section** 横截面 perpendicular to an axis, its volume is the integral of the cross-sectional **area**:

$$V = \int_a^b A(x)\,dx.$$

For **square** cross sections with side equal to the region's height $s(x) = f(x) - g(x)$, use $A(x) = [s(x)]^2$; for a **rectangle** of height $k \cdot s(x)$, use $A(x) = k\,[s(x)]^2$. The exam phrase "the base of a solid... cross sections are squares/rectangles" signals exactly this.

Volumes with Cross Sections: Triangles and Semicircles

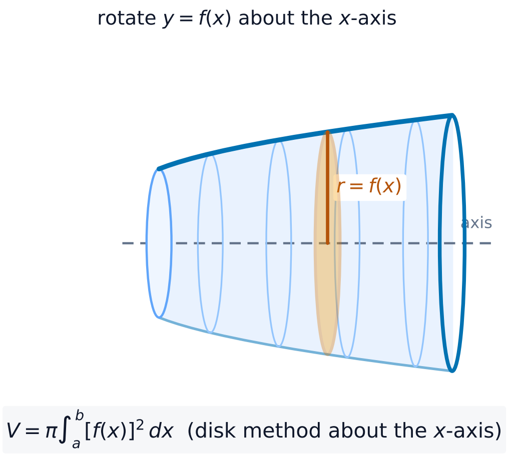
Same method, different area formula: for an equilateral **triangle** 三角形 of side s, $A = \frac{\sqrt{3}}{4}s^2$; for a **semicircle** 半圆 of diameter s, $A = \frac{\pi}{8}s^2$. Set the shape's key length equal to the region's slice length $s(x)$, then integrate $A(x)$.

Volume with the Disc Method (About the x- or y-Axis)

Revolving a region around an axis makes a **solid of revolution** 旋转体. If the region touches the axis, each slice is a **disc** 圆盘 of radius r = the function value:

$$V = \pi \int_a^b [r(x)]^2 \, dx.$$

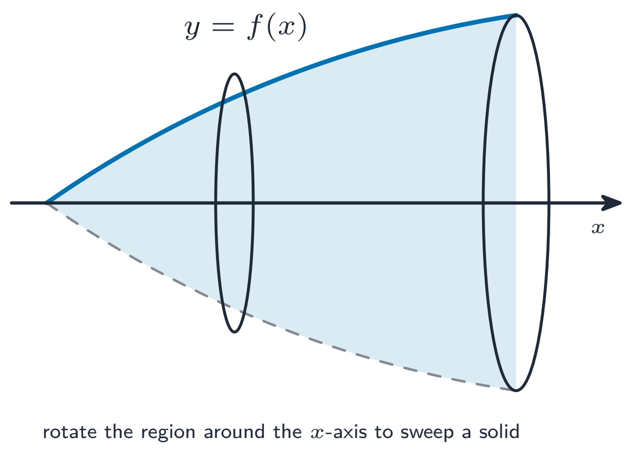
Around the x-axis, $r = f(x)$ and integrate in x; around the y-axis, express x as a function of y and integrate in y.



The disc method: rotating y=f(x) about the axis sweeps out disks of radius f(x)

Worked example. Revolve the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ about the x-axis. Each disc has radius $r = \sqrt{x}$, so

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \pi \left[\frac{x^2}{2} \right]_0^4 = 8\pi.$$



Rotating a region about an axis sweeps out a solid of revolution

Volume with the Disc Method (About Other Axes)

Around any horizontal or vertical line $y = k$ or $x = k$, the radius is the **distance** from the curve to that line, e.g. $r(x) = |f(x) - k|$. Set up the radius carefully, then use $V = \pi \int r^2$.

Volume with the Washer Method

When the region does **not** touch the axis, each slice is a **washer** 垫圈 (a ring) with an outer radius R and inner radius r:

$$V = \pi \int_a^b \left([R]^2 - [r]^2 \right) dx.$$

R is the distance from the axis to the **farther** boundary and r to the **nearer** one. Around a line other than an axis, both radii are measured as distances to that line – a very common exam variation (e.g. revolving about $y = -2$).

Exam tips

- Area between curves is $\int(\text{top} - \text{bottom})\,dx$ — find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f\,dx$.
- Accumulated change is $\int$ of a rate: total = initial value + $\int_a^b(\text{rate})\,dt$.
- Integration means "adding up infinitely many tiny pieces" — set up the integrand as one thin slice.