Integration and Accumulation of Change

AP Calculus AB

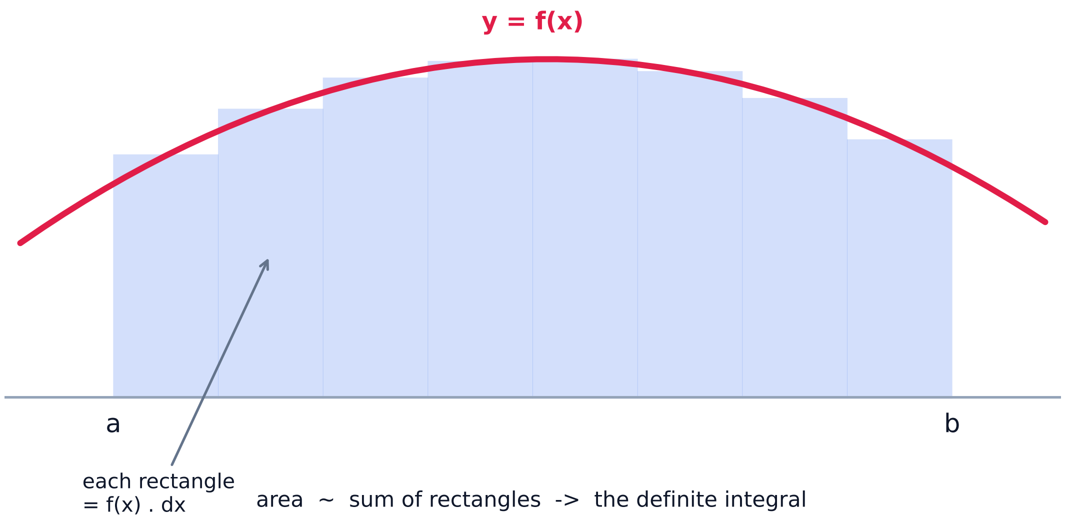
Exploring Accumulations of Change

Differentiation found rates. Integration runs the idea in reverse: given a **rate of change**, it finds the **accumulated change** 累积变化. The key picture: the **area** between the graph of a rate function and the x-axis gives the total accumulation.

- If the rate is positive over an interval, the accumulated change is positive; if negative, negative. Area below the axis counts as negative.
- Simple regions (triangles, rectangles) can be found with **geometry** 几何.
- **Units:** the area's unit is the rate's unit times the input's unit. A rate in vehicles-per-hour times hours gives **vehicles**.

Approximating Areas with Riemann Sums

When exact area is hard, approximate it with a **Riemann sum** 黎曼和–split the interval into subintervals and add up rectangle (or trapezoid) areas. The four standard estimates:



Each rectangle has area $f(x)\,\Delta x$; adding them estimates the area, and as the strips narrow the sum approaches the definite integral.

- **Left** Riemann sum –rectangle height from the **left** endpoint of each subinterval.
- **Right** Riemann sum –height from the **right** endpoint.
- **Midpoint** Riemann sum –height from the **midpoint** 中点.
- **Trapezoidal** sum 梯形法–average the two endpoint heights (a trapezoid).

Subintervals may be **uniform** (equal width) or **nonuniform** –read widths from the table.

Over- or underestimate? Judge from the behavior of the function: for an **increasing** function, a left sum underestimates and a right sum overestimates; a **trapezoidal** sum overestimates when the function is **concave up** and underestimates when concave down. Exam parts ask you to state which and why.

Worked example. A table gives $f(0) = 3,\ f(2) = 5,\ f(4) = 8,\ f(6) = 9$. Estimate $\int_0^6 f(x)\,dx$ with a **right** Riemann sum of three equal subintervals ($\Delta x = 2$). Use the right endpoint of each strip:

$$2\big(f(2) + f(4) + f(6)\big) = 2(5 + 8 + 9) = 44.$$

Since f is increasing, this right sum is an **overestimate**; the left sum $2(3 + 5 + 8) = 32$ would be an underestimate.

Riemann Sums and Integral Notation

As the subinterval widths shrink to zero, the Riemann sum approaches an exact value – the **definite integral** 定积分:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

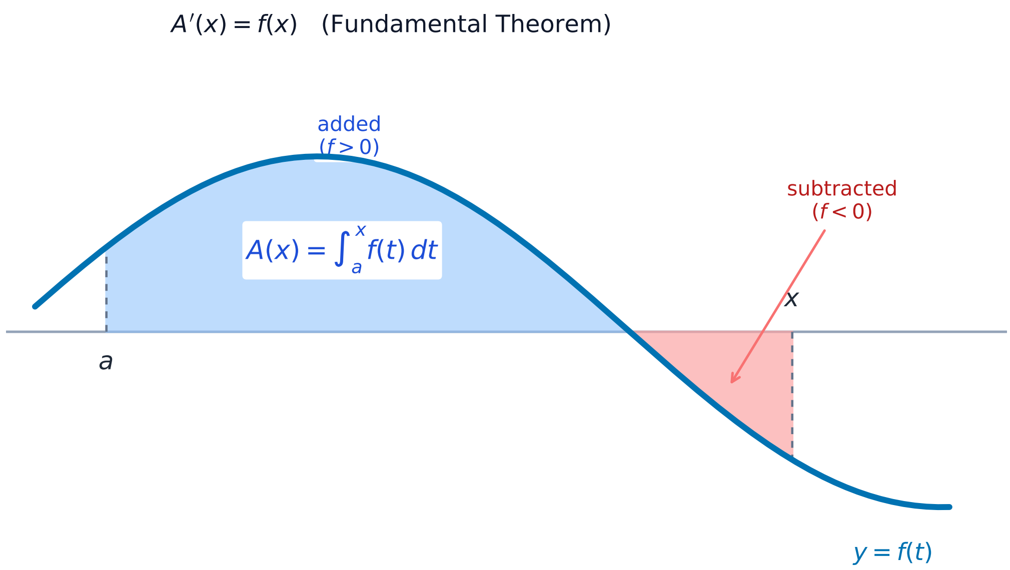
Here Δx_i is the width of the ith subinterval and x_i^* a point inside it. So a definite integral *is* the limit of a Riemann sum, and you should be able to translate each into the other.

The Fundamental Theorem of Calculus and Accumulation Functions

A definite integral with a variable upper limit defines a new **accumulation function** 累积函数. The **Fundamental Theorem of Calculus** 微积分基本定理 (first part) says differentiation undoes this accumulation: if f is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

So if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$ and $g''(x) = f'(x)$. This is the engine behind the very common "let $g(x) = \int_a^x f(t)\,dt$" questions.



The accumulation function adds signed area; the FTC says its derivative is f

Behavior of Accumulation Functions

Because $g'(x) = f(x)$, everything from Unit 5 applies to an accumulation function using the graph of f:

- g is **increasing** where $f > 0$ and **decreasing** where $f < 0$;
- g has a local extremum where f **crosses zero** (with a sign change);
- g is **concave up** where f is **increasing**; inflection points of g occur where f has a local extremum.

To get a *value* of g, compute the signed area: $g(x) = \int_a^x f(t)\,dt$, adding areas above the axis and subtracting areas below.

Properties of Definite Integrals

These properties simplify computation and appear constantly:

$$\int_a^a f = 0, \qquad \int_b^a f = -\int_a^b f, \qquad \int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g,$$

$$\int_a^b k\,f = k\int_a^b f, \qquad \int_a^c f + \int_c^b f = \int_a^b f.$$

The last (splitting at an interior point c) lets you build a total integral from pieces read off a graph.

The Fundamental Theorem and Evaluating Integrals

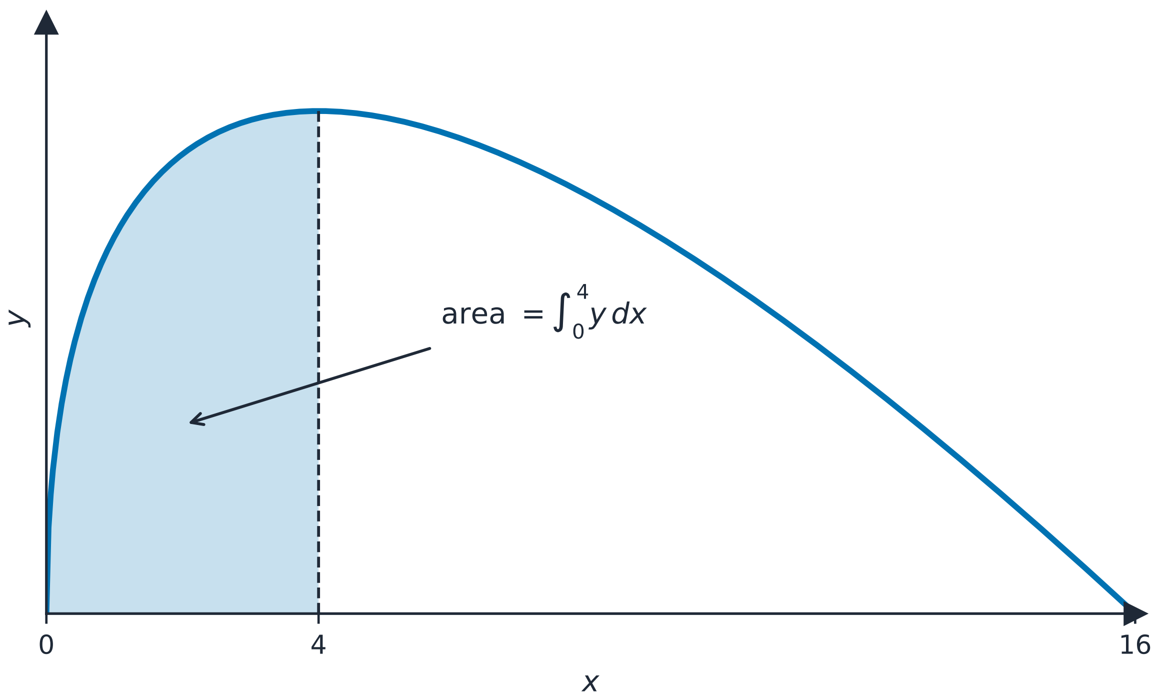
The second part of the Fundamental Theorem evaluates a definite integral using an **antiderivative 原函数**. If $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

So: find any antiderivative F, then subtract its values at the two limits. This is how most exact integrals are computed. It also gives the **net change** view: $\int_a^b g'(t)\,dt = g(b) - g(a)$, so a starting value plus accumulated change gives a later value, e.g. $g(5) = g(0) + \int_0^5 g'(t)\,dt$.

Worked example. Evaluate $\int_1^3 (2x+1)\,dx$. An antiderivative is $F(x) = x^2 + x$, so

$$\int_1^3 (2x+1)\,dx = F(3) - F(1) = (9+3) - (1+1) = 12 - 2 = 10.$$



A definite integral is the signed area between the curve and the x-axis

Antiderivatives and Indefinite Integrals

An **indefinite integral 不定积分** is the family of all antiderivatives, written with a **constant of integration 积分常数**:

$$\int f(x)\,dx = F(x) + C.$$

Reverse each derivative rule to build the basic antiderivatives:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1), \qquad \int \frac{1}{x}\,dx = \ln|x| + C, \qquad \int e^x\,dx = e^x + C,$$

$$\int \cos x\,dx = \sin x + C, \qquad \int \sin x\,dx = -\cos x + C.$$

Integrating Using Substitution

u**-substitution** 换元积分法 reverses the chain rule. Choose an inside function $u = g(x)$, so $du = g'(x)\,dx$, and rewrite the integral entirely in u:

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du.$$

Look for a function *and its derivative* both present. For a **definite** integral, either change the limits to u-values or convert back to x before substituting the original limits.

Worked example. Evaluate $\int 2x\cos(x^2)\,dx$. The inside function is $u = x^2$, whose derivative $2x\,dx = du$ is present, so

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

Spotting that $2x$ is exactly $\dfrac{du}{dx}$ is the whole trick.

Long Division and Completing the Square

Two algebraic set-up moves let more integrals fit the basic forms: **polynomial long division** 多项式长除法 when the top degree is $\geq$ the bottom degree of a rational function, and **completing the square** 配方法 to turn a quadratic denominator into a form that integrates to an arctangent or logarithm.

Selecting Techniques for Antidifferentiation

A skill topic: match the integral to a method. Try a basic antiderivative first; look for a u-substitution (an inside function whose derivative is also present); use algebra (division, completing the square, splitting a fraction) to reshape the integrand into a standard form. Naming the structure first prevents wasted effort.

Exam tips

- Integration is antidifferentiation; use the **power rule** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and don't forget the $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f'(x)\,dx = f(b) - f(a)$, and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.
- A definite integral is a signed area (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.