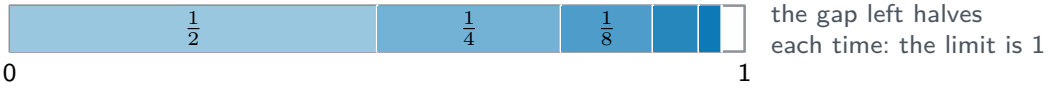
Limits and Continuity

AP Calculus AB

Introducing Calculus: Can Change Occur at an Instant?

Why it matters. About 2,400 years ago the Greek thinker Zeno argued that a runner can never finish a race: first she must cover half the distance, then half of what is left, then half of that, for ever. Infinitely many steps, he said, cannot be finished. The answer took more than two thousand years: the distances $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ get as close to 1 as we like, so their limit is exactly 1. Newton and Leibniz built calculus on this idea in the 1660s-1680s, and Augustin-Louis Cauchy gave it a careful definition in 1821. Every derivative and integral in this course is a limit.



Zeno's halves add up to a limit of 1.

Calculus is the mathematics of **change** 变化 and of **accumulation** 累积. It answers two big questions: how fast is something changing *right now*, and how much has piled up *so far*? Unit 1 builds the one tool both questions rest on – the **limit** 极限.

Start with a puzzle. A car's speedometer reads 60 km/h. What does that mean *at a single instant* 瞬间? Speed is distance over time. But at one instant no time passes and no distance is covered, so the fraction looks like $\frac{0}{0}$ – undefined.

- The **average rate of change** 平均变化率 uses a whole *interval* 区间: the change in one quantity divided by the change in another. It divides by zero, and so is undefined, when the change in the input would be zero.
- The **instantaneous rate of change** 瞬时变化率 is what we want *at a point*. It is the value the average rate **approaches** 趋近 as the interval shrinks toward zero length.

The clever move is not to plug in zero (undefined), but to watch what the average rate *approaches* as the interval gets smaller and smaller. That approaching value is a **limit**. So calculus lets us describe change at an instant – as a limit of average rates over ever-shorter intervals. This one idea powers the **derivative** 导数 (Unit 2) and, run in reverse, the **integral** 积分 (Unit 6). Everything else in this unit defines limits carefully and computes them reliably.

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Defining Limits and Using Limit Notation

Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c – **but not equal to** c. We write

$$\lim_{x \to c} f(x) = R$$

and read it: "the limit of $f(x)$, as x approaches c, equals R."



A smooth bridge curve: limits describe the value a graph approaches as we zoom in

The last words are the heart of a limit: it describes the **behavior** 行为 of f *near* c, not the value *at* c. The function may be undefined at c, or defined but equal to something else – the limit does not care.

A limit can be shown in three ways: **graphically** 用图象, **numerically** 用数值 (a table), and **analytically** 用解析式 (algebra). Learning to move between these representations is a core skill.

*(Note: the epsilon-delta definition of a limit is **not** tested on the AP Exam, so this handout does not use it.)*

Estimating Limit Values from Graphs

A graph is often the fastest way to read a limit. To find $\lim_{x \to c} f(x)$, run your finger along the curve toward $x = c$ from each side and ask: what height is the curve heading for?

- Trace from the **left** (inputs smaller than c): this gives the **left-hand limit** 左极限, $\lim_{x \to c^-} f(x)$.
- Trace from the **right** (inputs larger than c): this gives the **right-hand limit** 右极限, $\lim_{x \to c^+} f(x)$.

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- These are the **one-sided limits** 单侧极限. If both head to the *same* height R, then the two-sided limit exists and $\lim_{x\to c} f(x) = R$.

Crucially, **ignore the point itself**. Graphs mark the difference between the limit and the value:

- An **open circle** 空心圆 marks a height the curve *approaches* but does not reach – a "hole" 空洞.
- A **closed circle** 实心圆 marks the actual value $f(c)$.

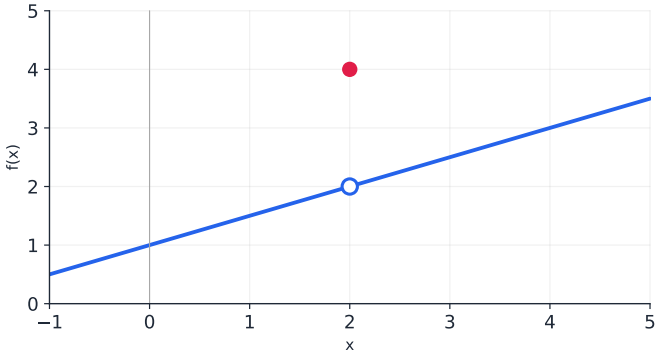
A limit **does not exist** (often written **DNE**) when the two sides disagree (a **jump** 跳跃), when the function is **unbounded** 无界 (grows without limit), or when it **oscillates** 振荡 near c with fluctuations that do not shrink toward one value. Oscillation alone does not prevent a limit: $x\sin(1/x) \to 0$ as $x \to 0$. For example:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x\to 0} \frac{|x|}{x} \text{ DNE}, \qquad \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \text{ DNE}.$$

Watch the **scale** 比例 of a graph: a zoomed-out picture can hide important behavior near a point, so confirm with algebra when you can.

■ Worked comparison: nearby values and one point

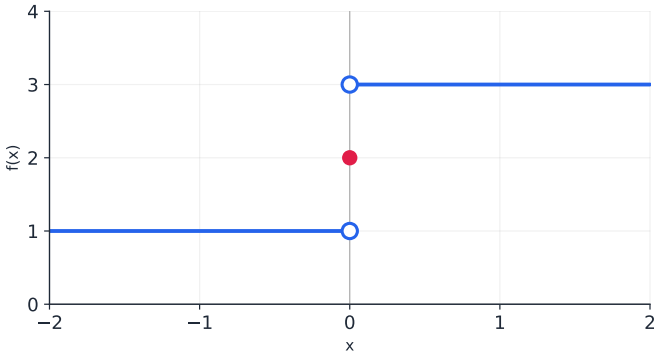
In this teacher-created graph, read the approaching heights before the filled point.



A curve with an open point at (2,2) and a filled point at (2,4).

Both one-sided limits at 2 equal 2, so the two-sided limit is 2. The filled point gives $f(2) = 4$. This is a removable discontinuity: set $f(2) = 2$ to make the function continuous.

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One-sided heights 1 and 3 at 0, with a separate filled point at height 2.

Here the left-hand limit at 0 is 1, the right-hand limit is 3, and $f(0) = 2$. The unequal one-sided limits mean the two-sided limit does not exist. No choice of $f(0)$ changes the nearby branches; the jump cannot be removed by changing one point.

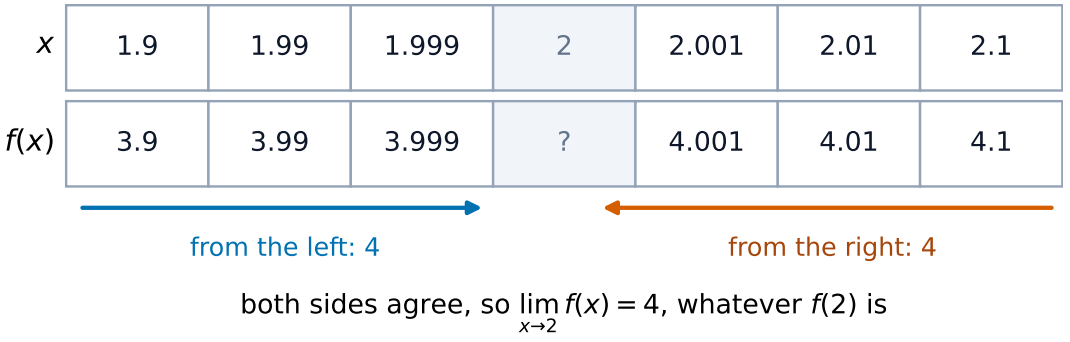
Estimating Limit Values from Tables

When you have data or a formula but no picture, a **table** 表格 of values estimates a limit numerically. Choose inputs that creep toward c from both sides and watch the outputs.

For example, to estimate $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$ (which is $\frac{0}{0}$ at $x = 2$):

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4, so we estimate the limit is 4. A table only *suggests* a value – it is a numerical estimate, not a proof.



Values from both sides close in on 4, so the limit is 4 whatever f(2) is

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Determining Limits Using Algebraic Properties of Limits

Most limits are found **analytically** using **limit theorems** 极限定理. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, the limit of a combination is the same combination of the limits:

- **Sum / difference:** $\lim\limits_{x\to c}\big[f(x)\pm g(x)\big] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$
- **Product:** $\lim\limits_{x\to c}\big[f(x)\,g(x)\big] = \lim\limits_{x\to c} f(x)\cdot \lim\limits_{x\to c} g(x)$
- **Quotient:** $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided the bottom limit is not 0.
- **Composite** 复合函数: if g is continuous at $\lim_{x\to c} f(x)$, then $\lim\limits_{x\to c} g\big(f(x)\big) = g\Big(\lim\limits_{x\to c} f(x)\Big)$.

The practical rule: for a function built from polynomials, roots, and the like, first try **direct substitution** 直接代入 – put $x = c$ in. If you get a real number, that is the limit. One-sided limits obey the same theorems, read from one direction only.

Determining Limits Using Algebraic Manipulation

Direct substitution sometimes gives the **indeterminate form** 未定式 $\frac{0}{0}$. This does **not** mean the limit fails – it means you must rewrite the function into an **equivalent form** 等价形式 that removes the trouble, then substitute. Three standard moves:

- **Factor and cancel** 因式分解并约分 a **rational function** 有理函数. Example: $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$.
- Multiply by the **conjugate** 共轭 to simplify a **radical** 根式. Example: $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x} = \lim\limits_{x\to 0}\dfrac{x}{x\big(\sqrt{x+1}+1\big)} = \dfrac{1}{2}$.
- **Use alternate forms** of trigonometric functions (identities) to simplify.

The cancelled factor is why the original graph had a hole: the two functions agree everywhere except at $x = c$, so they share the same limit there.

Selecting Procedures for Determining Limits

This is a skill topic, not new content: choose the right tool for the limit in front of you.

1. **Try direct substitution first.** A real answer means you are done.
2. Getting $\frac{0}{0}$? **Rewrite** – factor and cancel, or use the conjugate, or a trig identity – then substitute.
3. A **non-zero number over** 0 (like $\frac{5}{0}$)? The limit is infinite or DNE – check the sign from each side (see vertical asymptotes below).
4. As $x \to \pm\infty$? Compare the **dominant** 主导 terms (see limits at infinity).

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5. Trapped between two functions? The **squeeze theorem** may apply.

Determining Limits Using the Squeeze Theorem

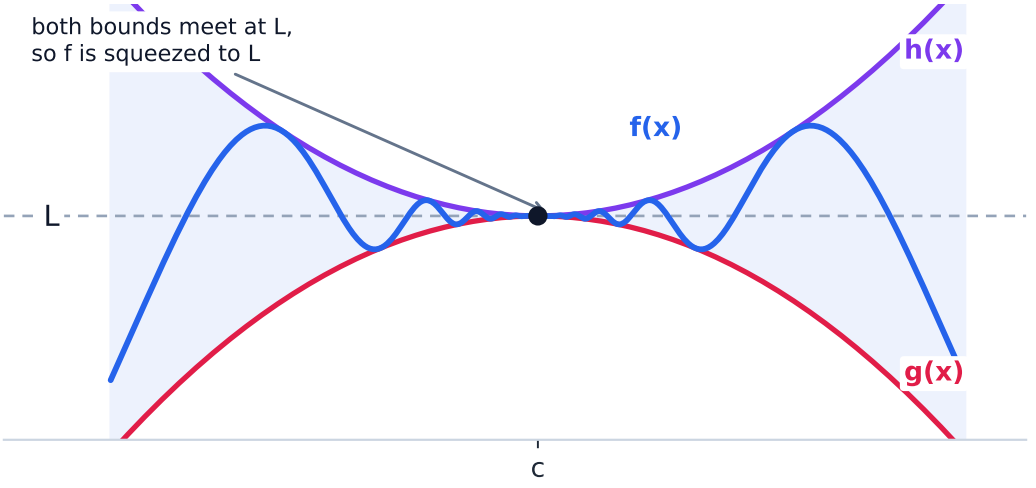
The **squeeze theorem** 夹逼定理 (also called the sandwich theorem) finds a limit by trapping the function between two others. If $g(x) \le f(x) \le h(x)$ near c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then f is squeezed to the same place: $\lim_{x \to c} f(x) = L$.

The two famous results proved this way, both used throughout calculus, are:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$



if $g \le f \le h$ near c and $\lim g = \lim h = L$, then $\lim f = L$

The wiggly f is trapped between g and h. Because both bounds meet at L, f has nowhere to go but L.

Connecting Multiple Representations of Limits

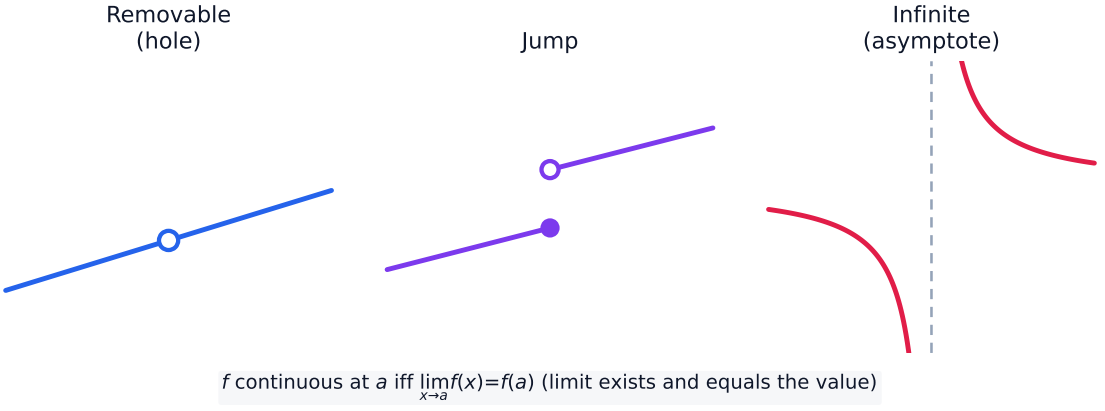
Another skill topic: the same limit lives in a graph, a table, and an algebraic form, and you should be able to translate between them. A graph shows the shape and any holes or jumps; a table gives numerical evidence; algebra gives an exact value and a reason. Strong answers use one representation to *confirm* another.

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Exploring Types of Discontinuities

A function is **discontinuous** 间断 at c when its graph "breaks" there. Three common types are:

- **Removable discontinuity** 可去间断 – a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** 跳跃间断 – the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** 无穷间断 – the function blows up to $\pm\infty$ at a **vertical asymptote** 垂直渐近线.



Three ways a graph can break: a removable hole, a jump, and an infinite (asymptote) discontinuity.

■ Worked comparison: classify and justify a repair

At 2, $A(x) = (x^2 - 4)/(x - 2)$ simplifies to $x + 2$ for $x \neq 2$. Its limit is 4; define $A(2) = 4$ to remove the hole. In contrast, $B(x) = 1$ for $x < 2$ and 3 for $x \geq 2$ has unequal finite one-sided limits 1 and 3. Changing $B(2)$ cannot remove its jump. For $C(x) = 1/(x - 2)$, the left side tends to $-\infty$ and the right to $+\infty$: an infinite discontinuity, with no finite limit to use as a repair.

Defining Continuity at a Point

Continuity is defined by a three-part test. A function f is **continuous** 连续 at $x = c$ exactly when all three hold:

$$\boxed{f(c) \text{ exists} \quad \text{and} \quad \lim_{x \to c} f(x) \text{ exists} \quad \text{and} \quad \lim_{x \to c} f(x) = f(c)}$$

In words: the point is there, the limit is there, and the two agree. If any one fails, f is discontinuous at c. This test is the backbone of nearly every continuity question, so learn it as a checklist.

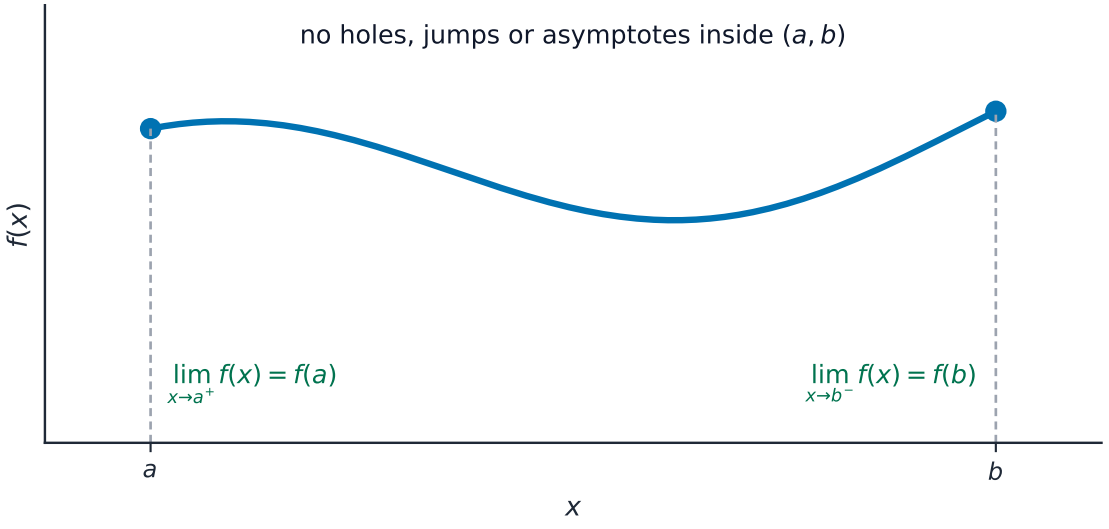
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Confirming Continuity over an Interval

A function is **continuous on an interval** 在区间上连续 if it is continuous at *every* point of that interval. You rarely check point by point, because whole families are continuous on their domains:

> **Polynomial, rational, power, exponential 指数, logarithmic 对数, and trigonometric 三角 functions are continuous at every point of their domains.**

So a rational function is continuous everywhere *except* where its denominator is zero; $\ln x$ is continuous for $x > 0$; and so on. Knowing this lets you declare continuity quickly and correctly.



Continuous on [a, b]: no breaks inside and the one-sided limits match at the ends

Removing Discontinuities

If the **limit exists** at a hole, the discontinuity is **removable**: redefine the function at that one point to equal the limit, and the graph is repaired. Formally, set the missing value to $\lim_{x \to c} f(x)$.

For a **piecewise-defined function** 分段函数, continuity at a boundary $x = c$ needs the two pieces to *meet*: the left piece's value, the right piece's value, and $f(c)$ must all be equal. This is a common exam setup – you solve for a **parameter** 参数 (an unknown constant) that makes the pieces match:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

> **Worked example.** Find the constant k that makes this function continuous at $x = 2$:

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$$f(x) = \begin{cases} kx + 1, & x < 2 \\ x^2 - k, & x \geq 2 \end{cases}$$

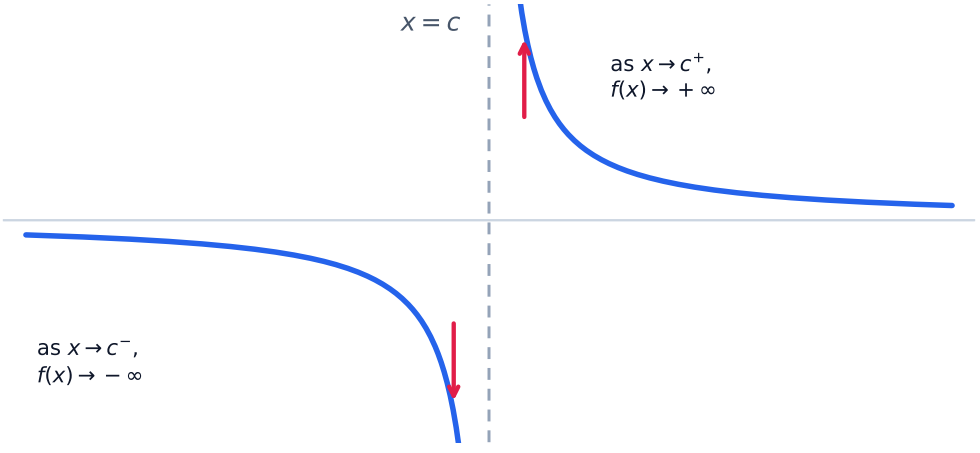
- Left-hand limit: $\lim_{x \to 2^-} (kx + 1) = 2k + 1$.
- Right-hand limit and value: $\lim_{x \to 2^+} (x^2 - k) = f(2) = 4 - k$.
- Set them equal: $2k + 1 = 4 - k$, so $3k = 3$ and $k = 1$.

Check: with $k = 1$ both pieces give 3 at $x = 2$, so all three conditions of the continuity test hold.

Connecting Infinite Limits and Vertical Asymptotes

The idea of a limit extends to **infinite limits** 无穷极限. When a function grows without bound near $x = c$, we write $\lim_{x \to c} f(x) = \pm\infty$. This describes a **vertical asymptote** at $x = c$: the graph hugs the vertical line $x = c$ and shoots off toward $\pm\infty$.

This happens where a **non-zero number is divided by something approaching** 0, such as at a zero of a denominator that does not cancel. Always check each side separately – the two sides can shoot **opposite** ways (one to $+\infty$, one to $-\infty$).



vertical asymptote $x = c$ when $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$

Near a vertical asymptote the graph hugs the line $x = c$. Here the left side falls to $-\infty$ and the right side climbs to $+\infty$, so the two-sided limit does not exist.

Worked example. Find $\lim_{x \to 1^-} \dfrac{x + 2}{x - 1}$ and $\lim_{x \to 1^+} \dfrac{x + 2}{x - 1}$.

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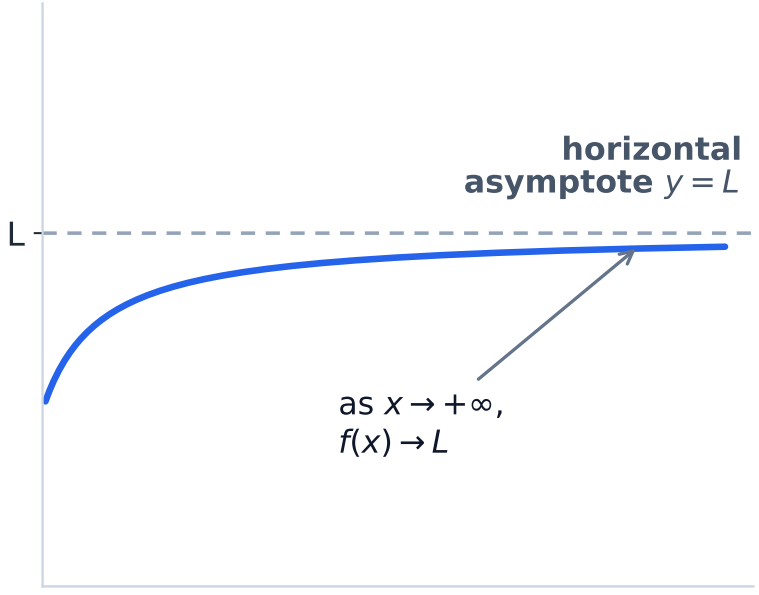
Substitution gives $\frac{3}{0}$: a non-zero number over 0, so each one-sided limit is infinite. Decide the sign from each side:

- For x just below 1 (try $x = 0.99$): top ≈ 3, bottom $= -0.01$, a small negative number. So $\lim\limits_{x\to 1^-} \dfrac{x+2}{x-1} = -\infty$.
- For x just above 1 (try $x = 1.01$): top ≈ 3, bottom $= +0.01$. So $\lim\limits_{x\to 1^+} \dfrac{x+2}{x-1} = +\infty$.

The line $x = 1$ is a vertical asymptote, and the two-sided limit does not exist.

Connecting Limits at Infinity and Horizontal Asymptotes

We can also let the **input** grow: **limits at infinity** 无穷远处的极限 describe the **end behavior** 末端行为 of a function as $x \to \pm\infty$. If the outputs settle toward a finite value L, then $y = L$ is a **horizontal asymptote** 水平渐近线.



As $x \to +\infty$ the curve flattens toward the line $y = L$. The horizontal asymptote records this end behavior.

For a rational function, compare the **degrees** 次数 of the top and bottom:

- top degree $<$ bottom degree $\Rightarrow$ limit is 0 (asymptote $y = 0$);
- top degree $=$ bottom degree $\Rightarrow$ limit is the **ratio of the leading coefficients** 首项系数之比;
- top degree $>$ bottom degree $\Rightarrow$ the function is unbounded (no horizontal asymptote).

Worked example. Find $\lim\limits_{x\to\infty} \dfrac{3x^2 - 5x}{6x^2 + 1}$.

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The degrees are equal (2 and 2), so the limit is the ratio of the leading coefficients. To show why, divide every term by x^2:

$$\lim_{x\to\infty}\frac{3x^2-5x}{6x^2+1}=\lim_{x\to\infty}\frac{3-\frac{5}{x}}{6+\frac{1}{x^2}}=\frac{3-0}{6+0}=\frac{1}{2}.$$

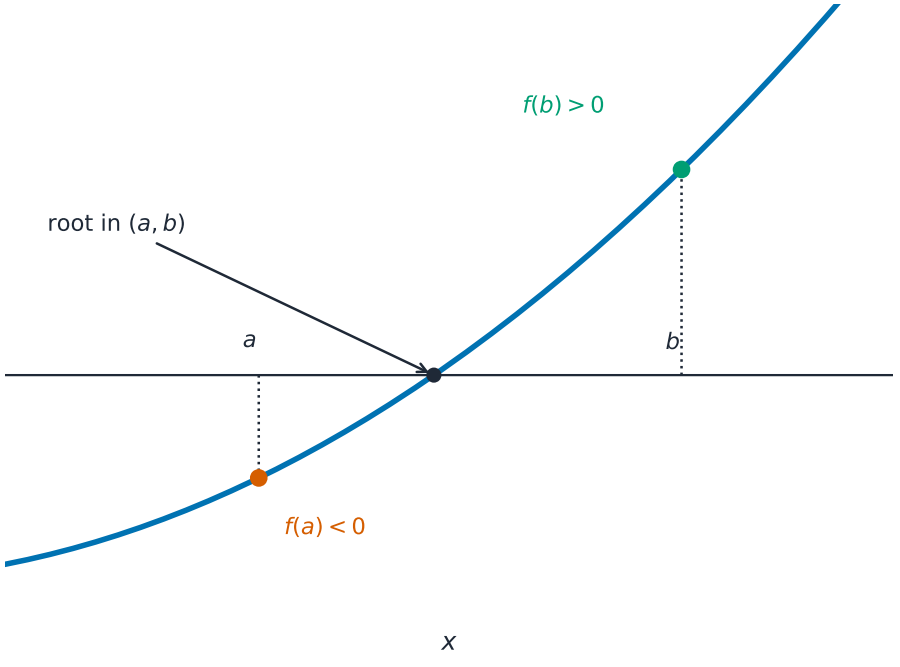
So $y=\frac{1}{2}$ is a horizontal asymptote.

More generally, we compare the **relative magnitudes** 相对大小 (relative growth rates) of functions: far out, an exponential beats any polynomial, and a polynomial beats any logarithm. On the exam, "as $t\to\infty$, which quantity is larger/where does the rate settle?" is answered with a limit at infinity.

Working with the Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem** 介值定理 is an **existence theorem** 存在性定理 – it guarantees a value exists without telling you where:

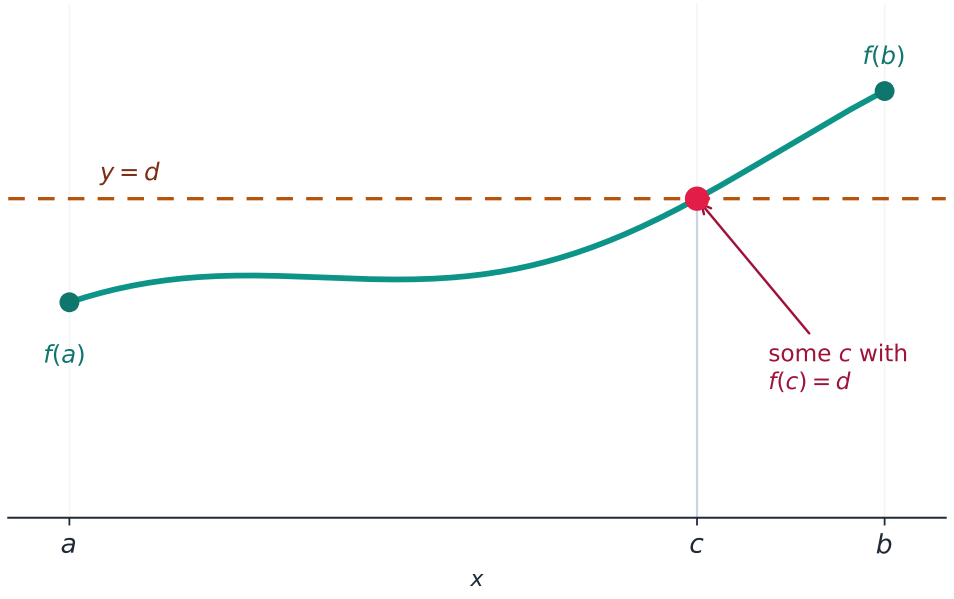
> If f is **continuous** on the closed interval $[a,b]$, and d is any number **between** $f(a)$ and $f(b)$, then there is **at least one** number c in (a,b) with $f(c)=d$.



Opposite signs of f(a) and f(b) trap a root between a and b

An unbroken curve cannot skip a height between its endpoints – it must pass through every one.

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A continuous curve from $(a, f(a))$ to $(b, f(b))$ must cross every height d in between at least once.

Exam skill – how to justify with the IVT. These questions appear almost every year (for example, "Must there be a value c with $R(c) = 155$?" or "Is there a time when $r'(t) = -6$?"). A full-credit justification has three moves:

1. **State continuity.** Say the function is continuous on $[a, b]$ (often because it is differentiable, or given continuous).
2. **Show d is trapped.** Compute the two endpoint values and show the target d lies **between** them, e.g. $f(a) < d < f(b)$.
3. **Conclude by name.** "By the Intermediate Value Theorem, there is a c in (a, b) with $f(c) = d$."

Skipping the continuity statement, or not showing d is between the endpoints, loses the point – the theorem *requires* both conditions.

> **Worked example.** Show that $f(x) = x^3 - x - 1$ has a zero between $x = 1$ and $x = 2$.

1. f is a polynomial, so it is continuous on $[1, 2]$.
2. $f(1) = 1 - 1 - 1 = -1$ and $f(2) = 8 - 2 - 1 = 5$, so $f(1) < 0 < f(2)$.
3. By the Intermediate Value Theorem, there is a c in $(1, 2)$ with $f(c) = 0$.

The theorem says nothing about *where* c is or how many such values there are (here $c \approx 1.32$, found with a calculator).

Common mistakes with the IVT.

- Using it on a function that is not continuous on the whole interval: $g(x) = \frac{1}{x}$ has $g(-1) = -1$ and $g(1) = 1$, but g is never 0, because g breaks at $x = 0$.

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- Reading it backwards: a continuous f can reach a height *outside* $[f(a), f(b)]$ too, and can reach d more than once. The theorem only promises *at least one* c for each d between the endpoint values.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ – the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes at denominator zeros that remain after common factors are cancelled; check the one-sided limits.

Common mistakes.

- Saying a limit does not exist because $f(a)$ is undefined. A hole in the graph does not stop the limit from existing.
- Stopping at $\frac{0}{0}$ and writing "0" or "undefined". $\frac{0}{0}$ is a signal to factor, rationalise or simplify, not an answer.
- Using the IVT without stating that the function is **continuous** on the closed interval - on the exam, the justification earns no credit without it.
- Writing $\lim_{x\to a} f(x) = \infty$ and then calling the limit a number. An infinite limit describes unbounded behaviour (a vertical asymptote); the limit does not exist as a real number.
- Finding a horizontal asymptote by substituting a large number instead of comparing the leading terms.

Practice. Questions 3 and 4 follow the style of recent free-response parts (IVT from a table; the squeeze theorem with two bounding functions).

1. Find $\displaystyle\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$.
2. Find $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$.
3. A continuous function f has $f(1) = 4$, $f(3) = -2$ and $f(5) = 6$. What is the least number of solutions that $f(x) = 0$ must have on $1 \le x \le 5$? Justify your answer.
4. For all x near 1, $2x + 1 \le k(x) \le x^2 + 2$. Find $\displaystyle\lim_{x\to 1} k(x)$, and say what else you need to know to conclude that k is continuous at $x = 1$.

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5. Let $f(x) = \dfrac{2x^2 - 8}{x^2 - x - 2}$. Find every vertical asymptote, horizontal asymptote and removable discontinuity.
6. Find k so that $f(x) = x^2 + k$ for $x < 2$ and $f(x) = 3x - 1$ for $x \geq 2$ is continuous at $x = 2$.

Answers.

1. Factor: $\dfrac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$, so the limit is 6.
2. $\dfrac{\sin 3x}{x} = 3 \cdot \dfrac{\sin 3x}{3x} \to 3 \cdot 1 = 3$.
3. At least two. f is continuous on $[1, 3]$ and $[3, 5]$; $f(1) = 4 > 0 > -2 = f(3)$, and $f(3) = -2 < 0 < 6 = f(5)$, so by the IVT there is a zero in $(1, 3)$ and another in $(3, 5)$.
4. Both bounds equal 3 at $x = 1$ (and $x^2 + 2 - (2x + 1) = (x - 1)^2 \geq 0$), so by the squeeze theorem $\lim_{x \to 1} k(x) = 3$. k is continuous at 1 only if also $k(1) = 3$.
5. $f(x) = \dfrac{2(x-2)(x+2)}{(x-2)(x+1)}$. Removable discontinuity (hole) at $x = 2$, where the limit is $\frac{2 \cdot 4}{3} = \frac{8}{3}$; vertical asymptote $x = -1$; horizontal asymptote $y = 2$ (ratio of leading coefficients).
6. Left limit $4 + k$ must equal $f(2) = 5$, so $k = 1$.

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