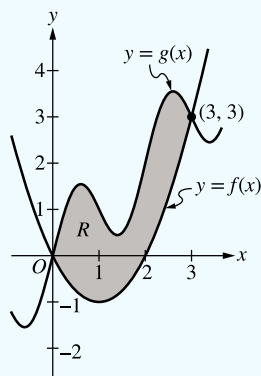


Part A (AB): Graphing calculator required**Question 2****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The shaded region R is bounded by the graphs of the functions f and g , where $f(x) = x^2 - 2x$ and $g(x) = x + \sin(\pi x)$, as shown in the figure.



(Note: Your calculator should be in radian mode.)

Model Solution**Scoring**

A Find the area of R . Show the setup for your calculations.

$$\int_0^3 (g(x) - f(x)) \, dx$$

Form of integrand **Point 1 (P1)**

$$= 5.136620$$

Answer **Point 2 (P2)**

The area is 5.137 (or 5.136).

Scoring Notes for Part A

- P1** is earned for a response that presents an integrand of $g(x) - f(x)$, $|g(x) - f(x)|$, $f(x) - g(x)$, or $|f(x) - g(x)|$ in a definite integral, with or without the differential dx .
- P1** could also be earned for a difference of definite integrals with integrands $g(x)$ and $f(x)$.
- P2** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point.
- Incorrect communication between the integral and the correct answer is treated as scratch work and is not considered in scoring.
 - $\int_0^3 (f(x) - g(x)) \, dx = -5.137$ so the area is 5.137.
Note: This response earns **P1** for the integral. It also earns **P2** for the correct answer.
 - $\int_0^3 (f(x) - g(x)) \, dx = 5.137$
Note: This response earns **P1** for the integral. It also earns **P2** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)
- The exact answer is $\frac{4 + 9\pi}{2\pi}$.

- B** Region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x -axis is a rectangle with height x and base in region R . Find the volume of the solid. Show the setup for your calculations.

$\int_0^3 x(g(x) - f(x)) \, dx$	Form of integrand	Point 3 (P3)
$= 7.704930$	Answer	Point 4 (P4)

The volume of the solid is 7.705 (or 7.704).

Scoring Notes for Part B

- P3** is earned for a definite integral with an integrand presented as a product of two nonconstant factors, with one of the factors equal to x , $g(x) - f(x)$, or $f(x) - g(x)$.
- The presence or absence of the differential dx will not be considered in scoring **P3** or **P4**.
- P4** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.
- Incorrect or unclear communication between the integral and the correct answer is treated as scratch work and is not considered in scoring. For example:
 - $\int_0^3 x(f(x) - g(x)) \, dx = -7.705$ so the volume is 7.705.
Note: This response earns **P3** for the integral. It also earns **P4** for the correct answer.
 - $\int_0^3 x(f(x) - g(x)) \, dx = 7.705$
Note: This response earns **P3** for the integral. It also earns **P4** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)
- The exact answer is $\frac{12 + 27\pi}{4\pi}$.

- C** Write, but do not evaluate, an integral expression for the volume of the solid generated when the region R is rotated about the horizontal line $y = -2$.

$\text{Volume} = \pi \int_0^3 ((g(x) - (-2))^2 - (f(x) - (-2))^2) \, dx$	Form of integrand	Point 5 (P5)
	Integrand	Point 6 (P6)
	Limits, constant, and differential	Point 7 (P7)

Scoring Notes for Part C

- P5** is earned for a definite integral with an integrand of $R^2 - r^2$ or $|R^2 - r^2|$, where one of $\{R, r\}$ is correct or a difference between g and a nonzero constant, and the other is correct or a difference between f and a nonzero constant.
- P6** is earned for the integral $\int_0^3 ((g(x) + 2)^2 - (f(x) + 2)^2) \, dx$, $\int_0^3 ((f(x) + 2)^2 - (g(x) + 2)^2) \, dx$, or a mathematically equivalent expression.
- Note **P5** and **P6** could be earned for a difference of definite integrals.
- A response that presents an integral expression that does not include the constant π is eligible for **P5** and **P6** but does not earn **P7**.
- To be eligible for **P7**, a response must have earned **P5**.
- A response that reverses the difference of squares must resolve the reversal with either the constant or the limits of integration AND include the differential to earn **P7**. For example:
 - A response of $-\pi \int_0^3 ((f(x) + 2)^2 - (g(x) + 2)^2) \, dx$ or $\pi \int_3^0 ((f(x) + 2)^2 - (g(x) + 2)^2) \, dx$ earns **P5**, **P6**, and **P7**.
 - A response of $\pi \int_0^3 ((f(x) + 2)^2 - (g(x) + 2)^2) \, dx$ earns **P5** and **P6** but does not earn **P7**.
- A response of only $\pi \int_0^3 ((g(x) - (-2))^2 - (f(x) - (-2))^2) \, dx$ earns **P5**, **P6**, and **P7**.

- D** It can be shown that $g'(x) = 1 + \pi \cos(\pi x)$. Find the value of x , for $0 < x < 1$, at which the line tangent to the graph of f is parallel to the line tangent to the graph of g .

$f'(x) = g'(x) \Rightarrow 2x - 2 = 1 + \pi \cos(\pi x)$	$f'(x) = g'(x)$	Point 8 (P8)
$\Rightarrow x = 0.675819$	Answer	Point 9 (P9)

The lines tangent to the graphs of f and g are parallel at $x = 0.676$ (or 0.675).

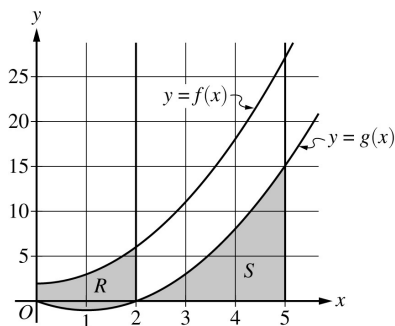
Scoring Notes for Part D

- P8** is earned for a general statement, such as $f'(x) = g'(x)$, or for any correct equation formed by substituting $2x - 2$ for $f'(x)$, $1 + \pi \cos(\pi x)$ for $g'(x)$, or both.
- P9** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

Part B (AB): Graphing calculator not allowed**Question 6****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



The functions f and g are defined by $f(x) = x^2 + 2$ and $g(x) = x^2 - 2x$, as shown in the graph.

Model Solution**Scoring**

- (a) Let R be the region bounded by the graphs of f and g , from $x = 0$ to $x = 2$, as shown in the graph. Write, but do not evaluate, an integral expression that gives the area of region R .

$$\text{Area} = \int_0^2 (f(x) - g(x)) \, dx$$

Integrand	1 point
Answer	1 point

Scoring notes:

- The first point is earned for a response that presents an integrand of $f(x) - g(x)$, $|f(x) - g(x)|$, $g(x) - f(x)$, or $|g(x) - f(x)|$ in one or more definite integrals.
- The first point could also be earned for a difference of definite integrals with integrands $f(x)$ and $g(x)$.
- The second point is earned only for one or more integrals equivalent to $\int_0^2 (f(x) - g(x)) \, dx$, such as $\int_0^2 f(x) \, dx - \int_0^2 g(x) \, dx$, $\int_0^2 |f(x) - g(x)| \, dx$, $-\int_0^2 (g(x) - f(x)) \, dx$, or $\int_0^2 |g(x) - f(x)| \, dx$.
 - Note: $\int_0^2 f(x) \, dx + \left| \int_0^2 g(x) \, dx \right|$ would earn both points.

Total for part (a) 2 points

- (b) Let S be the region bounded by the graph of g and the x -axis, from $x = 2$ to $x = 5$, as shown in the graph. Region S is the base of a solid. For this solid, at each x the cross section perpendicular to the x -axis is a rectangle with height equal to half its base in region S . Find the volume of the solid. Show the work that leads to your answer.

Volume = $\int_2^5 \frac{1}{2}(g(x))^2 \, dx = \int_2^5 \frac{1}{2}(x^2 - 2x)^2 \, dx$	Integrand	1 point
$= \frac{1}{2} \int_2^5 (x^4 - 4x^3 + 4x^2) \, dx$	Limits	1 point
$= \frac{1}{2} \left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3} \right]_2^5$	Antiderivative	1 point
$= \frac{1}{2} \left[\left(\frac{5^5}{5} - 5^4 + \frac{500}{3} \right) - \left(\frac{32}{5} - 16 + \frac{32}{3} \right) \right]$	Answer	1 point
$= \frac{1}{2} \left(\frac{500}{3} - \frac{16}{15} \right) = \frac{414}{5}$		

Scoring notes:

- The first point is earned only by a response with an integrand of the form $k(g(x))^2$ in a definite integral, where k is any nonzero constant.
- The second point is earned for limits of $x = 2$ and $x = 5$ in a definite integral with an integrand of the form $a(x) \cdot g(x)$ for any nonzero factor $a(x)$.
- To earn the third point a response must provide a correct antiderivative of $k(x^2 - 2x)^n$ for some integer $n \geq 2$.
- The fourth point is earned only for a numeric answer equivalent to $\frac{1}{2} \left(\frac{500}{3} - \frac{16}{15} \right)$.
- Special case: A response of Volume = $\int_2^5 \frac{1}{2}(f(x))^2 \, dx$ or Volume = $\int_2^5 \frac{1}{2}(x^2 + 2)^2 \, dx$ earns the first 2 points and is not eligible for the last 2 points.

Total for part (b) 4 points

- (c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when region S , as described in part (b), is rotated about the horizontal line $y = 20$.

Volume $= \pi \int_2^5 [(20^2) - (20 - g(x))^2] dx$	Form of integrand	1 point
$= \pi \int_2^5 [400 - (20 - g(x))^2] dx$	Integrand	1 point
$= \pi \int_2^5 [400 - (20 - (x^2 - 2x))^2] dx$	Limits and constant	1 point

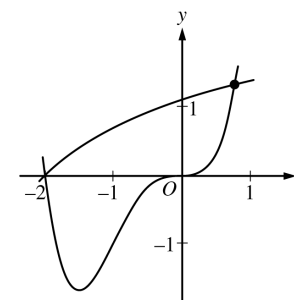
Scoring notes:

- The first point is earned for a response that presents an integrand of $20^2 - (20 - g(x))^2$, $20^2 - (g(x) - 20)^2$, $(20 - g(x))^2 - 20^2$, $(g(x) - 20)^2 - 20^2$, or any mathematically equivalent expression, in one or more definite integrals.
- The second point is earned only for an integrand mathematically equivalent to $20^2 - (20 - g(x))^2$ or $20^2 - (g(x) - 20)^2$ in a definite integral. Note that $\int_a^b |(20 - g(x))^2 - 400| dx$ or $|\int_a^b ((20 - g(x))^2 - 400) dx|$ earns the first 2 points.
- The integral may be split into two integrals.
 - For example, Volume $= \pi \int_2^5 (20^2) dx - \pi \int_2^5 (20 - g(x))^2 dx$ or
Volume $= \pi \cdot 20^2 \cdot 3 - \pi \int_2^5 (20 - g(x))^2 dx$.
- A response that presents an allowable integrand involving $g(x)$, but continues and makes an error in using the expression for $g(x)$, does not earn the second point.
 - For example, $\pi \int_2^5 [20^2 - (20 - g(x))^2] dx = \pi \int_2^5 [20^2 - (20 - x^2 - 2x)^2] dx$ earns the first point but does not earn the second point.
- To be eligible for the third point a response must have earned at least 1 of the first 2 points or must have presented an integrand involving $g(x)$ of the form $R^2 - r^2$ in a definite integral.
- The third point is earned only by a definite integral including the constant π and limits $x = 2$ to $x = 5$. A response that presents any other constant or limits (including $x = 5$ to $x = 2$, except in the note below) does not earn the third point.
 - Note: $\pi \int_5^2 [(20 - g(x))^2 - 400] dx$ would earn all three points.
- Special case: A response of $\pi \int_2^5 [400 - (20 - f(x))^2] dx$ or equivalent earns 2 of the 3 points.

Total for part (c) 3 points**Total for question 6 9 points****Part A (AB): Graphing calculator required****Question 2****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Let f and g be the functions defined by $f(x) = \ln(x + 3)$ and $g(x) = x^4 + 2x^3$. The graphs of f and g , shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.

Model Solution	Scoring
(a) Find the area of the region enclosed by the graphs of f and g .	
$\ln(x + 3) = x^4 + 2x^3 \Rightarrow x = -2, x = B = 0.781975$	Integrand 1 point
$\int_{-2}^B (f(x) - g(x)) dx = 3.603548$	Limits of integration 1 point
The area of the region is 3.604 (or 3.603).	Answer 1 point

Scoring notes:

- Other forms of the integrand in a definite integral, e.g., $|f(x) - g(x)|$, $|g(x) - f(x)|$, or $g(x) - f(x)$, earn the first point.
- To earn the second point, the response must have a lower limit of -2 and an upper limit expressed as either the letter B with no value attached, or a number that is correct to the number of digits presented, with at least one and up to three decimal places.
 - Case 1: If the response did not earn the second point because of an incorrect value of B , $0 < B < 1$, but used a lower limit of -2 , the response earns the third point only for a consistent answer.
 - Case 2: If the response did not earn the second point because the lower limit used was $x = 0$, but the response used a correct upper limit of B , the response earns the third point for a consistent answer of 0.708 (or 0.707).
 - Case 3: If a response uses any other incorrect limits it does not earn the second or third points.
- A response containing the integrand $g(x) - f(x)$ must interpret the value of the resulting integral correctly to earn the third point. For example, the following response earns all 3 points:
 $\int_{-2}^B (g(x) - f(x)) dx = -3.604$ so the area is 3.604 . However, the response
 “Area = $\int_{-2}^B (g(x) - f(x)) dx = 3.604$ ” presents an untrue statement and earns the first and second points but not the third point.
- A response must earn the first point in order to be eligible for the third point. If the response has earned the second point, then only the correct answer will earn the third point.
- Instructions for scoring a response that presents an integrand of $\ln(x+3) - x^4 + 2x^3$ and the correct answer are shown in the “Global Special Case” after part (d).

Total for part (a) 3 points

- (b) For $-2 \leq x \leq B$, let $h(x)$ be the vertical distance between the graphs of f and g . Is h increasing or decreasing at $x = -0.5$? Give a reason for your answer.

$h(x) = f(x) - g(x)$	Considers $h'(-0.5)$	1 point
$h'(x) = f'(x) - g'(x)$	– OR –	
$h'(-0.5) = f'(-0.5) - g'(-0.5) = -0.6$ (or -0.599)	$f'(x) - g'(x)$	
Since $h'(-0.5) < 0$, h is decreasing at $x = -0.5$.	Answer with reason	1 point

Scoring notes:

- The response need not present the value of $h'(-0.5)$. The last line earns both points. However, if a value is presented it must be correct for the digits reported up to three decimal places.
- A response that reports an incorrect value of $h'(-0.5)$ earns only the first point.
- A response that presents only $h'(x)$ does not earn either point.
- The only response that earns the second point for concluding “ h is increasing” is described in the “Global Special Case” provided after part (d).
- A response that compares the values of $f'(x)$ and $g'(x)$ at $x = -0.5$ earns the first point and is eligible for the second point. This comparison can be made symbolically or verbally; for example, the response “the rate of change of $f(x)$ is less than the rate of change of $g(x)$ at $x = -0.5$ ” earns the first point.

Total for part (b) 2 points

- (c) The region enclosed by the graphs of f and g is the base of a solid. Cross sections of the solid taken perpendicular to the x -axis are squares. Find the volume of the solid.

$\int_{-2}^8 (f(x) - g(x))^2 dx = 5.340102$	Integrand	1 point
The volume of the solid is 5.340.	Answer	1 point

Scoring notes:

- The first point is earned for an integrand of $k(f(x) - g(x))^2$ or its equivalent with $k \neq 0$ in any definite integral. If $k \neq 1$, then the response is not eligible for the second point.
- A response that does not earn the first point is ineligible to earn the second point, with the following exceptions:
 - A response which has a presentation error in the integrand (for example, mismatched or missing parentheses, misplaced exponent) does not earn the first point but would earn the second point for the correct answer. A response which has a presentation error in the integrand and which reports an incorrect answer earns no points.
 - A response that presents an integrand of $(\ln(x+3) - x^4 + 2x^3)^2$. Scoring instructions for this case are provided in the “Global Special Case” after part (d).
- A response that uses incorrect limits is only eligible for the second point, provided the limits are imported from part (a) in Case 1 or Case 2. In both of these situations, the second point is earned only for answers consistent with the imported limits.

Total for part (c) 2 points

- (d) A vertical line in the xy -plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of 7 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$.

The cross section has area $A(x) = (f(x) - g(x))^2$.	$\frac{dA}{dx} \cdot \frac{dx}{dt}$	1 point
$\frac{d}{dt}[A(x)] = \frac{dA}{dx} \cdot \frac{dx}{dt}$		
$\left. \frac{d}{dt}[A(x)] \right _{x=-0.5} = A'(-0.5) \cdot 7 = -9.271842$	Answer	1 point
At $x = -0.5$, the area of the cross section above the line is changing at a rate of -9.272 (or -9.271) square units per second.		

Scoring notes:

- The first point may be earned by presenting $\frac{dA}{dx} \cdot \frac{dx}{dt}$, $A' \cdot \frac{dx}{dt}$, $A'(x) \cdot \frac{dx}{dt}$, $A'(x) \cdot x'$, $A'(-0.5) \cdot 7$, or $k \cdot 7$, where k is a declared value of $A'(-0.5)$, or any equivalent expression, including $2(f(x) - g(x))(f'(x) - g'(x)) \frac{dx}{dt}$.
- If a response defines $f(x) - g(x)$ as a function in parts (b) or (c) (for example, $h(x) = f(x) - g(x)$), then a correct expression for $\frac{dA}{dt}$ (for example, $2h \frac{dh}{dt}$) earns the first point.
- A response that imports a function $A(x)$ declared in part (c) is eligible for both points (the answer must be consistent with the imported function $A(x)$).
- A response that presents an incorrect function for $A(x)$ that is not imported from part (c) is eligible only for the first point.
- Except when $A(x)$ is imported from part (c), the second point is earned only for the correct answer.
- A response that does not earn the first point is ineligible to earn the second point except in the special case noted below.

Total for part (d) 2 points**Total for question 2 9 points**

Global Special Case: A response may incorrectly simplify $f(x) - g(x)$ to $j(x) = \ln(x+3) - x^4 + 2x^3$ instead of $\ln(x+3) - x^4 - 2x^3$. Because this question is calculator active, a response with this incorrect simplification may nevertheless present correct answers.

- In any part of the question, a response that starts correctly by using $f(x) - g(x)$, then presents $j(x)$, is eligible for all points in that part.
- The first time a response implicitly presents $f(x) - g(x)$ as $j(x) = \ln(x+3) - x^4 + 2x^3$ (with no explicit connection) in any part of this question, the response loses a point. The response is then eligible for all remaining points for a correct or consistent answer.
- In part (a) the consistent answer using $j(x)$ is negative and will not earn the third point.
- In part (b) the consistent answer using $j(x)$ is that $j'(-0.5) = 2.4 > 0$, so h is increasing at $x = -0.5$.
- In part (c) the consistent answer using $j(x)$ is 252.187 (or 252.188).
- In part (d) the consistent answer using $j(x)$ is 20.287.

AP[®] Calculus AB

Scoring Guidelines

2019 SCORING GUIDELINES

Question 1

(a) $\int_0^5 E(t) dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

(b) $\frac{1}{5-0} \int_0^5 L(t) dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

- (c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$$E(t) - L(t) = 0 \Rightarrow t = 6.20356$$

$E(t) - L(t) > 0$ for $0 \leq t < 6.20356$, and $E(t) - L(t) < 0$ for $6.20356 < t \leq 8$. Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$$A(t) = \int_0^t (E(s) - L(s)) ds$$

$$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

2 : $\begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

2019 SCORING GUIDELINES

Question 2

- (a)
- v_P
- is differentiable
- $\Rightarrow v_P$
- is continuous on
- $0.3 \leq t \leq 2.8$
- .

$$\frac{v_P(2.8) - v_P(0.3)}{2.8 - 0.3} = \frac{55 - 55}{2.5} = 0$$

By the Mean Value Theorem, there is a value c , $0.3 < c < 2.8$, such that $v_P'(c) = 0$.

— OR —

v_P is differentiable $\Rightarrow v_P$ is continuous on $0.3 \leq t \leq 2.8$.

By the Extreme Value Theorem, v_P has a minimum on $[0.3, 2.8]$.

$$v_P(0.3) = 55 > -29 = v_P(1.7) \text{ and } v_P(1.7) = -29 < 55 = v_P(2.8).$$

Thus v_P has a minimum on the interval $(0.3, 2.8)$.

Because v_P is differentiable, $v_P'(t)$ must equal 0 at this minimum.

$$\begin{aligned} \text{(b)} \quad \int_0^{2.8} v_P(t) dt &\approx 0.3 \left(\frac{v_P(0) + v_P(0.3)}{2} \right) + 1.4 \left(\frac{v_P(0.3) + v_P(1.7)}{2} \right) \\ &\quad + 1.1 \left(\frac{v_P(1.7) + v_P(2.8)}{2} \right) \\ &= 0.3 \left(\frac{0 + 55}{2} \right) + 1.4 \left(\frac{55 + (-29)}{2} \right) + 1.1 \left(\frac{-29 + 55}{2} \right) \\ &= 40.75 \end{aligned}$$

- (c)
- $v_Q(t) = 60 \Rightarrow t = A = 1.866181$
- or
- $t = B = 3.519174$

$$v_Q(t) \geq 60 \text{ for } A \leq t \leq B$$

$$\int_A^B |v_Q(t)| dt = 106.108754$$

The distance traveled by particle Q during the interval $A \leq t \leq B$ is 106.109 (or 106.108) meters.

- (d) From part (b), the position of particle
- P
- at time
- $t = 2.8$
- is

$$x_P(2.8) = \int_0^{2.8} v_P(t) dt \approx 40.75.$$

$$x_Q(2.8) = x_Q(0) + \int_0^{2.8} v_Q(t) dt = -90 + 135.937653 = 45.937653$$

Therefore at time $t = 2.8$, particles P and Q are approximately $45.937653 - 40.75 = 5.188$ (or 5.187) meters apart.

$$2 : \begin{cases} 1 : v_P(2.8) - v_P(0.3) = 0 \\ 1 : \text{justification, using} \\ \quad \text{Mean Value Theorem} \end{cases}$$

— OR —

$$2 : \begin{cases} 1 : v_P(0.3) > v_P(1.7) \\ \quad \text{and } v_P(1.7) < v_P(2.8) \\ 1 : \text{justification, using} \\ \quad \text{Extreme Value Theorem} \end{cases}$$

1 : answer, using trapezoidal sum

$$3 : \begin{cases} 1 : \text{interval} \\ 1 : \text{definite integral} \\ 1 : \text{distance} \end{cases}$$

$$3 : \begin{cases} 1 : \int_0^{2.8} v_Q(t) dt \\ 1 : \text{position of particle } Q \\ 1 : \text{answer} \end{cases}$$

2019 SCORING GUIDELINES

Question 3

$$\begin{aligned} \text{(a)} \quad \int_{-6}^5 f(x) dx &= \int_{-6}^{-2} f(x) dx + \int_{-2}^5 f(x) dx \\ &\Rightarrow 7 = \int_{-6}^{-2} f(x) dx + 2 + \left(9 - \frac{9\pi}{4} \right) \\ &\Rightarrow \int_{-6}^{-2} f(x) dx = 7 - \left(11 - \frac{9\pi}{4} \right) = \frac{9\pi}{4} - 4 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_3^5 (2f'(x) + 4) dx &= 2 \int_3^5 f'(x) dx + \int_3^5 4 dx \\ &= 2(f(5) - f(3)) + 4(5 - 3) \\ &= 2(0 - (3 - \sqrt{5})) + 8 \\ &= 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5} \end{aligned}$$

— OR —

$$\begin{aligned} \int_3^5 (2f'(x) + 4) dx &= [2f(x) + 4x]_{x=3}^{x=5} \\ &= (2f(5) + 20) - (2f(3) + 12) \\ &= (2 \cdot 0 + 20) - (2(3 - \sqrt{5}) + 12) \\ &= 2 + 2\sqrt{5} \end{aligned}$$

- (c)
- $g'(x) = f(x) = 0 \Rightarrow x = -1, x = \frac{1}{2}, x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$$\begin{aligned} \text{(d)} \quad \lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} &= \frac{10^1 - 3f'(1)}{f(1) - \arctan 1} \\ &= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}} \end{aligned}$$

$$3 : \begin{cases} 1 : \int_{-6}^5 f(x) dx = \int_{-6}^{-2} f(x) dx + \int_{-2}^5 f(x) dx \\ 1 : \int_{-2}^5 f(x) dx \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$$

1 : answer

2019 SCORING GUIDELINES

Question 4

(a) $V = \pi r^2 h = \pi(1)^2 h = \pi h$

$$\left. \frac{dV}{dt} \right|_{h=4} = \pi \left. \frac{dh}{dt} \right|_{h=4} = \pi \left(-\frac{1}{10} \sqrt{4} \right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

(b) $\frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10} \sqrt{h} \right) = \frac{1}{200}$

Because $\frac{d^2 h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

(c) $\frac{dh}{\sqrt{h}} = -\frac{1}{10} dt$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5} \right)^2$$

$$2 : \begin{cases} 1 : \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$$

$$3 : \begin{cases} 1 : \frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}} \\ 1 : \frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$$

$$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$$

Note: 0/4 if no separation of variables

Note: max 2/4 [1-1-0-0] if no constant of integration

2019 SCORING GUIDELINES

Question 5

$$\begin{aligned} \text{(a)} \quad \int_0^2 (h(x) - g(x)) dx &= \int_0^2 \left((6 - 2(x-1)^2) - \left(-2 + 3 \cos\left(\frac{\pi}{2}x\right) \right) \right) dx \\ &= \left[6x - \frac{2}{3}(x-1)^3 - \left(-2x + \frac{6}{\pi} \sin\left(\frac{\pi}{2}x\right) \right) \right]_{x=0}^{x=2} \\ &= \left(\left(12 - \frac{2}{3} \right) - (-4 + 0) \right) - \left(\left(0 + \frac{2}{3} \right) - (0 + 0) \right) \\ &= 12 - \frac{2}{3} + 4 - \frac{2}{3} = \frac{44}{3} \end{aligned}$$

The area of R is $\frac{44}{3}$.

$$\begin{aligned} \text{(b)} \quad \int_0^2 A(x) dx &= \int_0^2 \frac{1}{x+3} dx \\ &= \left[\ln(x+3) \right]_{x=0}^{x=2} = \ln 5 - \ln 3 \end{aligned}$$

The volume of the solid is $\ln 5 - \ln 3$.

$$\text{(c)} \quad \pi \int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2 \right) dx$$

$$4 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative of } 3 \cos\left(\frac{\pi}{2}x\right) \\ 1 : \text{antiderivative of remaining terms} \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{form of integrand} \\ 1 : \text{integrand} \end{cases}$$

2019 SCORING GUIDELINES

Question 6

(a) $h'(2) = \frac{2}{3}$

1 : answer

(b) $a'(x) = 9x^2h(x) + 3x^3h'(x)$

$$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$$

3 : $\begin{cases} 1 : \text{form of product rule} \\ 1 : a'(x) \\ 1 : a'(2) \end{cases}$ (c) Because h is differentiable, h is continuous, so $\lim_{x \rightarrow 2} h(x) = h(2) = 4$.

$$\text{Also, } \lim_{x \rightarrow 2} h(x) = \lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3}, \text{ so } \lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4.$$

$$\text{Because } \lim_{x \rightarrow 2} (x^2 - 4) = 0, \text{ we must also have } \lim_{x \rightarrow 2} (1 - (f(x))^3) = 0.$$

$$\text{Thus } \lim_{x \rightarrow 2} f(x) = 1.$$

$$\text{Because } f \text{ is differentiable, } f \text{ is continuous, so } f(2) = \lim_{x \rightarrow 2} f(x) = 1.$$

Also, because f is twice differentiable, f' is continuous, so

$$\lim_{x \rightarrow 2} f'(x) = f'(2) \text{ exists.}$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x \rightarrow 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$$

$$\text{Thus } f'(2) = -\frac{1}{3}.$$

4 : $\begin{cases} 1 : \lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4 \\ 1 : f(2) \\ 1 : \text{L'Hospital's Rule} \\ 1 : f'(2) \end{cases}$ (d) Because g and h are differentiable, g and h are continuous, so

$$\lim_{x \rightarrow 2} g(x) = g(2) = 4 \text{ and } \lim_{x \rightarrow 2} h(x) = h(2) = 4.$$

Because $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x \rightarrow 2} k(x) = 4$.

$$\text{Also, } 4 = g(2) \leq k(2) \leq h(2) = 4, \text{ so } k(2) = 4.$$

Thus k is continuous at $x = 2$.

1 : continuous with justification

AP[®] Calculus AB
2015 Scoring Guidelines

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2015 SCORING GUIDELINES

Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R , where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

- (a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \leq t \leq 8$?
- (b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.
- (c) At what time t , $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.
- (d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t) dt = 76.570$

2 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$

Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

2 : $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t , $0 \leq t \leq 8$, is $30 + \int_0^t [R(x) - D(x)] dx$.

$$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

3 : $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $30 + \int_0^w [R(t) - D(t)] dt = 50$

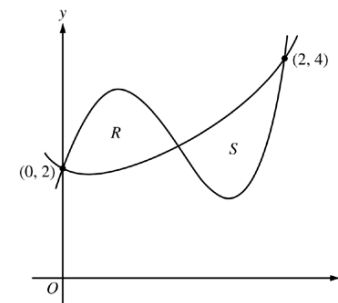
2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

2015 SCORING GUIDELINES

Question 2

Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2-2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

- (a) Find the sum of the areas of regions R and S .
- (b) Region S is the base of a solid whose cross sections perpendicular to the x -axis are squares. Find the volume of the solid.
- (c) Let h be the vertical distance between the graphs of f and g in region S . Find the rate at which h changes with respect to x when $x = 1.8$.



- (a) The graphs of $y = f(x)$ and $y = g(x)$ intersect in the first quadrant at the points $(0, 2)$, $(2, 4)$, and $(A, B) = (1.032832, 2.401108)$.

$$\begin{aligned} \text{Area} &= \int_0^A [g(x) - f(x)] dx + \int_A^2 [f(x) - g(x)] dx \\ &= 0.997427 + 1.006919 = 2.004 \end{aligned}$$

4 : $\begin{cases} 1 : \text{limits} \\ 2 : \text{integrands} \\ 1 : \text{answer} \end{cases}$

(b) $\text{Volume} = \int_A^2 [f(x) - g(x)]^2 dx = 1.283$

3 : $\begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\begin{aligned} h(x) &= f(x) - g(x) \\ h'(x) &= f'(x) - g'(x) \\ h'(1.8) &= f'(1.8) - g'(1.8) = -3.812 \text{ (or } -3.811) \end{aligned}$

2 : $\begin{cases} 1 : \text{considers } h' \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	-220	150

Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function v . Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

- (a) Use the data in the table to estimate the value of $v'(16)$.
- (b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)| dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)| dt$ using a right Riemann sum with the four subintervals indicated in the table.

- (c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

- (d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

(a) $v'(16) \approx \frac{240 - 200}{20 - 12} = 5$ meters/min²

1 : approximation

- (b) $\int_0^{40} |v(t)| dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \leq t \leq 40$ minutes.

3 : $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned} \int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters} \end{aligned}$$

- (c) Bob's acceleration is $B'(t) = 3t^2 - 12t$.

$$B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$$

2 : $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

- (d) Avg vel = $\frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$

$$\begin{aligned} &= \frac{1}{10} \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10} \\ &= \frac{1}{10} \left[\frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min} \end{aligned}$$

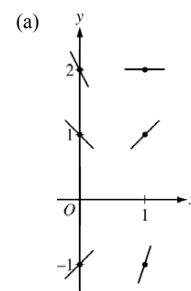
3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.
- (d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.



2 : $\begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.

Therefore, all solution curves are concave up in Quadrant II.

(c) $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

2 : $\begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

2 : $\begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

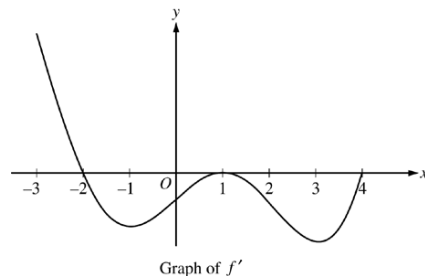
$$b = -m \Rightarrow b = -2$$

3 : $\begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 5

The figure above shows the graph of f' , the derivative of a twice-differentiable function f , on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x -axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.



- (a) Find all x -coordinates at which f has a relative maximum. Give a reason for your answer.
- (b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.
- (c) Find the x -coordinates of all points of inflection for the graph of f . Give a reason for your answer.
- (d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

- (a) $f'(x) = 0$ at $x = -2$, $x = 1$, and $x = 4$.
 $f'(x)$ changes from positive to negative at $x = -2$.
 Therefore, f has a relative maximum at $x = -2$.

2 : $\begin{cases} 1 : \text{identifies } x = -2 \\ 1 : \text{answer with reason} \end{cases}$

- (b) The graph of f is concave down and decreasing on the intervals $-2 < x < -1$ and $1 < x < 3$ because f' is decreasing and negative on these intervals.

2 : $\begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

- (c) The graph of f has a point of inflection at $x = -1$ and $x = 3$ because f' changes from decreasing to increasing at these points.

2 : $\begin{cases} 1 : \text{identifies } x = -1, 1, \text{ and } 3 \\ 1 : \text{reason} \end{cases}$

The graph of f has a point of inflection at $x = 1$ because f' changes from increasing to decreasing at this point.

- (d) $f(x) = 3 + \int_1^x f'(t) dt$

$$f(4) = 3 + \int_1^4 f'(t) dt = 3 + (-12) = -9$$

$$f(-2) = 3 + \int_1^{-2} f'(t) dt = 3 - \int_{-2}^1 f'(t) dt = 3 - (-9) = 12$$

3 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{expression for } f(x) \\ 1 : f(4) \text{ and } f(-2) \end{cases}$

2015 SCORING GUIDELINES

Question 6

Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

- (a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.
- (b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.
- (c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

(a) $\frac{dy}{dx}\bigg|_{(x,y)=(-1,1)} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{4}$

2 : $\begin{cases} 1 : \text{slope} \\ 1 : \text{equation for tangent line} \end{cases}$

An equation for the tangent line is $y = \frac{1}{4}(x + 1) + 1$.

(b) $3y^2 - x = 0 \Rightarrow x = 3y^2$

So, $y^3 - xy = 2 \Rightarrow y^3 - (3y^2)(y) = 2 \Rightarrow y = -1$

$(-1)^3 - x(-1) = 2 \Rightarrow x = 3$

The tangent line to the curve is vertical at the point $(3, -1)$.

3 : $\begin{cases} 1 : \text{sets } 3y^2 - x = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{coordinates} \end{cases}$

(c) $\frac{d^2y}{dx^2} = \frac{(3y^2 - x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$

$$\frac{d^2y}{dx^2}\bigg|_{(x,y)=(-1,1)} = \frac{(3 \cdot 1^2 - (-1)) \cdot \frac{1}{4} - 1 \cdot \left(6 \cdot 1 \cdot \frac{1}{4} - 1\right)}{(3 \cdot 1^2 - (-1))^2} = \frac{1 - \frac{1}{2}}{16} = \frac{1}{32}$$

4 : $\begin{cases} 2 : \text{implicit differentiation} \\ 1 : \text{substitution for } \frac{dy}{dx} \\ 1 : \text{answer} \end{cases}$