Part B (AB or BC): Graphing calculator not allowed

Question 3 — 9 points

General Scoring Notes

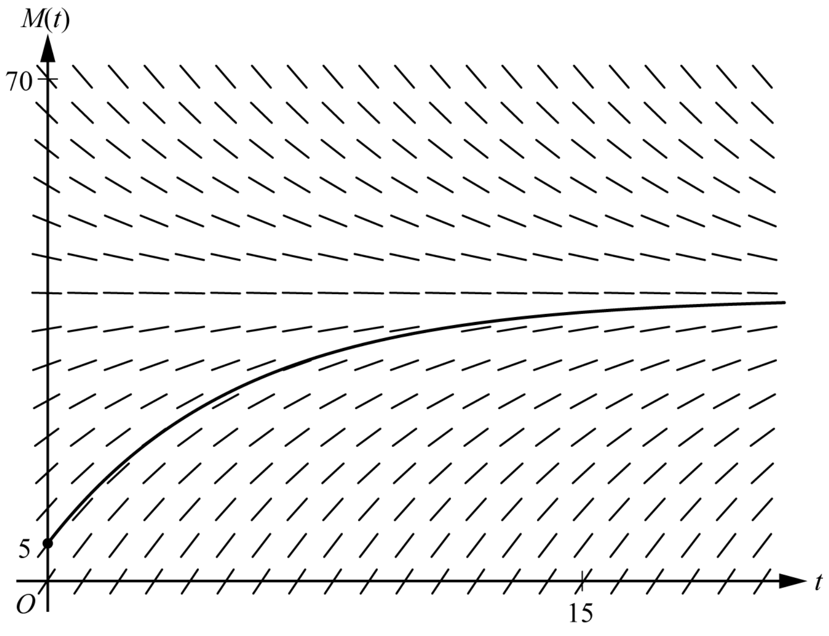
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($°C$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is $5°C$. It can be shown that $M(t) < 40$ for all values of t.

Model Solution	Scoring

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.



Solution curve — **1 point**

Scoring notes:

- The solution curve must pass through the point $(0, 5)$, extend reasonably close to the left and right edges of the rectangle, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must lie entirely below the horizontal line segments at $M = 40$.

Total for part (a) — 1 point

(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

$\left.\frac{dM}{dt}\right\|_{t=0} = \frac{1}{4}(40 - 5) = \frac{35}{4}$	$\left.\frac{dM}{dt}\right\|_{t=0}$	**1 point**
The tangent line equation is $y = 5 + \frac{35}{4}(t - 0)$. $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$ The temperature of the milk at time $t = 2$ minutes is approximately 22.5° Celsius.	Approximation	**1 point**

Scoring notes:

- The value of the slope may appear in a tangent line equation or approximation.
- A response of $5 + \frac{35}{4} \cdot 2$ is the minimal response to earn both points.
- A response of $\frac{1}{4}(40 - 5)$ earns the first point. If there are any subsequent errors in simplification, the response does not earn the second point.
- In order to earn the second point the response must present an approximation found by using a tangent line that:
  - passes through the point $(0, 5)$ and
  - has slope $\frac{35}{4}$ or a nonzero slope that is declared to be the value of $\frac{dM}{dt}$.
- An unsupported approximation does not earn the second point.
- The approximation need not be simplified, but the response does not earn the second point if the approximation is simplified incorrectly.

Total for part (b) 2 points

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

$\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right) = -\frac{1}{16}(40 - M)$	$\frac{d^2M}{dt^2}$	**1 point**
Because $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down. Therefore, the tangent line approximation of $M(2)$ is an overestimate.	Overestimate with reason	**1 point**

Scoring notes:

- The first point is earned for either $\frac{d^2M}{dt^2} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right)$ or $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$ (or equivalent). A response that presents any subsequent simplification error does not earn the second point.
- A response that presents an expression for $\frac{d^2M}{dt^2}$ in terms of $\frac{dM}{dt}$ but fails to continue to an expression in terms of M (i.e., $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt}$) does not earn the first point but is eligible for the second point.
- If the response presents an expression for $\frac{d^2M}{dt^2}$ that is incorrect, the response is eligible for the second point only if the expression is a nonconstant linear function that is negative for $5 < M < 40$.
  - Special case: A response that presents $\frac{d^2M}{dt^2} = \frac{1}{16}(40 - M)$ does not earn the first point but is eligible to earn the second point for a consistent answer and reason.
- To earn the second point a response must include $\frac{d^2M}{dt^2} < 0$, or $\frac{dM}{dt}$ is decreasing, or the graph of M is concave down, as well as the conclusion that the approximation is an overestimate.
- A response that presents an argument based on $\frac{d^2M}{dt^2}$ or concavity at a single point does not earn the second point.

Total for part (c) 2 points

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

$\frac{dM}{40 - M} = \frac{1}{4}dt$ $\int \frac{dM}{40 - M} = \int \frac{1}{4}\,dt$	Separates variables	**1 point**
$-\ln\lvert 40 - M\rvert = \frac{1}{4}t + C$	Finds antiderivatives	**1 point**
$-\ln\lvert 40 - 5\rvert = 0 + C \Rightarrow C = -\ln 35$ $M(t) < 40 \Rightarrow 40 - M > 0 \Rightarrow \lvert 40 - M\rvert = 40 - M$ $-\ln(40 - M) = \frac{1}{4}t - \ln 35$ $\ln(40 - M) = -\frac{1}{4}t + \ln 35$	Constant of integration and uses initial condition	**1 point**

$40 - M = 35e^{-t/4}$ $M = 40 - 35e^{-t/4}$	Solves for M	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 4 points.
- A response that presents an antiderivative of $-\ln(40 - M)$ without absolute value symbols is eligible for all 4 points.
- A response with no constant of integration can earn at most the first 2 points.
- A response is eligible for the third point only if it has earned the first 2 points.
    - Special Case: A response that presents $+\ln(40 - M) = \frac{t}{4} + C$ (or equivalent) does not earn the second point, is eligible for the third point, but not eligible for the fourth.
- An eligible response earns the third point by correctly including the constant of integration in an equation and substituting 0 for t and 5 for M.
- A response is eligible for the fourth point only if it has earned the first 3 points.
- A response earns the fourth point only for an answer of $M = 40 - 35e^{-t/4}$ or equivalent.

Total for part (d) **4 points**

Total for question 3 **9 points**

Part B (AB): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

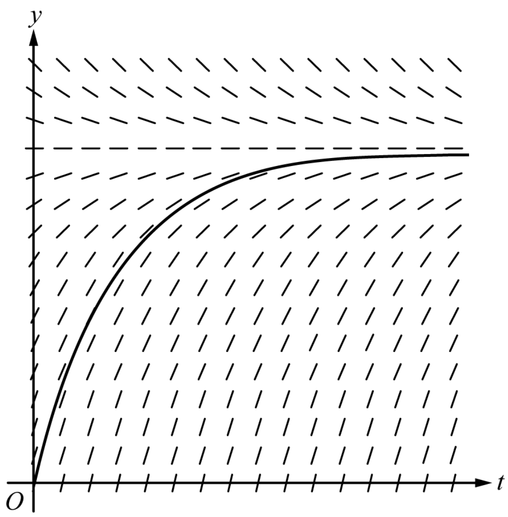
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

Model Solution	Scoring

(a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.

	Solution curve **1 point**

Scoring notes:

- To earn the point the solution curve must pass through the point $(0, 0)$, be generally increasing and concave down, and approach the horizontal asymptote from below as t increases. The point is not earned if two or more solution curves are presented.

Total for part (a) 1 point

(b) Using correct units, interpret the statement $\lim_{t \to \infty} A(t) = 12$ in the context of this problem.

Over time the amount of medication in the patient approaches 12 milligrams.	Interpretation	**1 point**

Scoring notes:

- To earn the point the interpretation must include "medication in the patient," "approaches 12," and units (milligrams), or their equivalents.

Total for part (b) 1 point

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ with initial condition $A(0) = 0$.

$\frac{dy}{dt} = \frac{12 - y}{3} \Rightarrow \frac{dy}{12 - y} = \frac{dt}{3}$	Separation of variables	**1 point**
$\int \frac{dy}{12 - y} = \int \frac{dt}{3} \Rightarrow -\ln\lvert 12 - y \rvert = \frac{t}{3} + C$	Antiderivatives	**1 point**
$\ln\lvert 12 - y \rvert = -\frac{t}{3} - C \Rightarrow \lvert 12 - y \rvert = e^{-t/3 - C}$ $\Rightarrow y = 12 + Ke^{-t/3}$ $0 = 12 + K \Rightarrow K = -12$	Constant of integration and uses initial condition	**1 point**
$y = A(t) = 12 - 12e^{-t/3}$	Solves for y	**1 point**

Scoring notes:

- A response of $\frac{dy}{12 - y} = 3\,dt$ is a bad separation and does not earn the first point. However, this response is eligible for the second and third points. It cannot earn the fourth point.
- Absolute value bars are not required in this part.
- A response that correctly separates to $\frac{3\,dy}{12 - y} = dt$ but then incorrectly simplifies to $\frac{dy}{4 - y} = dt$ earns the first point (for the initial correct separation), is eligible for the second point (for $-\ln\lvert 4 - y \rvert = t$, with or without $+C$), but is not eligible for the third or fourth points.
- $+\ln\lvert 12 - y \rvert = \frac{t}{3}$ (with or without $+C$) does not earn the second point and is not eligible for the fourth point; $+\ln\lvert 12 - y \rvert = \frac{t}{3} + C$ is eligible for the third point.
- In all other cases, the points are earned consecutively—the second point cannot be earned without the first, the third without the second, etc.

Total for part (c) 4 points

(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

$\frac{dy}{dt} = 3 - \frac{y}{t+2} \Rightarrow \frac{d^2y}{dt^2} = (-1)\frac{\frac{dy}{dt}(t+2) - y}{(t+2)^2}$	Quotient rule	**1 point**
$B'(1) = 3 - \frac{B(1)}{3} = 3 - \frac{2.5}{3} = \frac{6.5}{3}$ $B''(1) = -\frac{B'(1)\cdot 3 - B(1)}{3^2} = -\frac{6.5 - 2.5}{9} = -\frac{4}{9} < 0$	$B''(1) < 0$	**1 point**
The rate of change of the amount of medication is decreasing at time $t = 1$ because $B''(1) < 0$ and $\frac{d^2y}{dt^2}$ is continuous in an interval containing $t = 1$.	Answer with reason	**1 point**

Scoring notes:

- The first point is for correctly applying the quotient rule to $\frac{y}{t+2}$ or applying the product rule to $y(t+2)^{-1}$. Errors in differentiating the constant, 3, or handling the sign of the second term of $\frac{dy}{dt}$ will result in not earning the second point.
- The second point cannot be earned unless the second derivative $\frac{d^2y}{dt^2}$ is correct.
- For the second point it is sufficient to state the sign of $B''(1)$ is negative with supporting work. If a value is declared for $B''(1)$, it must be correct in order to earn the second point.
- Eligibility for the third point: An attempt at using the quotient rule (or product rule) to find $B''(1)$. In this case the third point will be earned for a consistent conclusion based on the declared value (or sign) of $B''(1)$.

Total for part (d) 3 points

Total for question 6 9 points