

Part B (AB): Graphing calculator not allowed**Question 6****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

	Model Solution	Scoring
A	Show that $\frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$.	
	$\frac{d}{dx}\left(y^3 - y^2 - y + \frac{1}{4}x^2\right) = \frac{d}{dx}(0)$ $\Rightarrow 3y^2 \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} - \frac{dy}{dx} + \frac{x}{2} = 0$	Implicit differentiation Point 1 (P1)
	$\Rightarrow (3y^2 - 2y - 1)\frac{dy}{dx} = -\frac{x}{2}$ $\Rightarrow \frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$	Verification Point 2 (P2)
Scoring Notes for Part A		

- P1** is earned only for the correct implicit differentiation of $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$. Responses may use alternative notations for $\frac{dy}{dx}$, such as y' .
- To be eligible for **P2**, a response must have earned **P1**.
- It is sufficient to present $(3y^2 - 2y - 1)\frac{dy}{dx} = -\frac{x}{2}$ to earn **P2**, provided there are no subsequent errors.

- B** There is a point P on the curve G near $(2, -1)$ with x -coordinate 1.6. Use the line tangent to the curve at $(2, -1)$ to approximate the y -coordinate of point P .

$$\left. \frac{dy}{dx} \right|_{(x, y)=(2, -1)} = \frac{-2}{2(3 + 2 - 1)} = -\frac{1}{4}$$

Slope of tangent line **Point 3 (P3)**

$$y \approx -1 - \frac{1}{4}(1.6 - 2) = -0.9$$

Tangent line approximation **Point 4 (P4)**

Scoring Notes for Part B

- A response can earn **P3** with $\left. \frac{dy}{dx} \right|_{(x, y)=(2, -1)} = -\frac{1}{4}$, $\frac{dy}{dx} = -\frac{1}{4}$, “slope is $-\frac{1}{4}$,” or equivalent.
- A response that presents a linear approximation with a slope of $-\frac{1}{4}$ also earns **P3**.
- A response that declares $\left. \frac{dy}{dx} \right|_{(x, y)=(2, -1)}$ (or the slope) equal to any nonzero value $k \neq -\frac{1}{4}$ does not earn **P3**. Such a response earns **P4** for a presented approximation mathematically equivalent to $-1 + k(-0.4)$.
- P4** cannot be earned with a linear approximation using a slope other than $-\frac{1}{4}$ if that slope has not been declared to be the value of $\left. \frac{dy}{dx} \right|_{(x, y)=(2, -1)}$.
- A response does not have to present the tangent line equation but must clearly demonstrate its use at $x = 1.6$ in finding the requested approximation to be eligible for **P4**.
- A response of $-1 - \frac{1}{4}(-0.4)$ earns both **P3** and **P4**.
- A response of $-1 - \frac{1}{4}(1.6 - 2)$ or equivalent earns **P4** (i.e., subsequent errors in simplification will not be considered in scoring for **P4**).

Note: An ambiguous response, such as $-1 - \frac{1}{4}(1.6 - 2)$, does not earn **P4** and therefore must go on to resolve the ambiguity with a correct final answer (e.g., -0.9) to earn **P4**.

- C** For $x > 0$ and $y > 0$, there is a point S on the curve G at which the line tangent to the curve at that point is vertical. Find the y -coordinate of point S . Show the work that leads to your answer.

For $x > 0$, the curve G has a vertical tangent line when
 $2(3y^2 - 2y - 1) = 0$.

Sets denominator equal to 0 **Point 5 (P5)**

$$2(3y^2 - 2y - 1) = 0 \Rightarrow 2(3y + 1)(y - 1) = 0$$

Because $y > 0$, it follows that $y = 1$.

Answer **Point 6 (P6)**

The line tangent to the curve is vertical at the point on the curve where $y = 1$.

Scoring Notes for Part C

- **P5** is earned with any of $2(3y^2 - 2y - 1) = 0$, $3y^2 - 2y - 1 = 0$, $2(3y \pm 1)(y \pm 1) = 0$, or $(3y \pm 1)(y \pm 1) = 0$.
- To be eligible for **P6**, a response must have earned **P5**.
- A response does not need to consider the numerator of $\frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$ to earn **P5** or **P6**; considering the denominator is sufficient.
- A response that states solutions of $y = -\frac{1}{3}$ and $y = 1$, but does not clearly identify that $y = 1$ is the only solution to the prompt, does not earn **P6**.
- In the presence of algebraic work to find the value of y , **P6** is earned only if the algebraic work is correct.
- A response of “ $2(3y^2 - 2y - 1) = 0$, $y = 1$ ” earns both **P5** and **P6**.

- D** A particle moves along the curve H defined by the equation $2xy + \ln y = 8$. At the instant when the particle is at the point $(4, 1)$, $\frac{dx}{dt} = 3$. Find $\frac{dy}{dt}$ at that instant. Show the work that leads to your answer.

$$\frac{d}{dt}(2xy + \ln y) = \frac{d}{dt}(8)$$

$$2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$$

Attempts implicit differentiation with respect to t **Point 7 (P7)**

$$2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$$
 Point 8 (P8)

$$2(3)(1) + 2(4)\frac{dy}{dt} + \frac{1}{1}\frac{dy}{dt} = 0$$

$$\Rightarrow 6 + 9\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{2}{3}$$

Answer **Point 9 (P9)**

Scoring Notes for Part D

- P7** is earned for implicitly differentiating $2xy + \ln y = 8$ with respect to t with at most one error.
- P8** is earned for an equation equivalent to $2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$.
- To be eligible for **P9**, a response must have earned **P7** and **P8**, with no errors in implicit differentiation.
- P9** is earned only for the value of $-\frac{2}{3}$.
- Alternate solution:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{d}{dx}(2xy + \ln y) = \frac{d}{dx}(8) \Rightarrow 2y + 2x\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2y}{2x + \frac{1}{y}}$$

$$\left. \frac{dy}{dx} \right|_{(x,y)=(4,1)} = \frac{-2(1)}{2(4) + \frac{1}{1}} = -\frac{2}{9}$$

$$\left. \frac{dy}{dt} \right|_{(x,y)=(4,1)} = \left. \frac{dy}{dx} \cdot \frac{dx}{dt} \right|_{(x,y)=(4,1)} = -\frac{2}{9} \cdot 3 = -\frac{2}{3}$$

- P7** is earned for an implicit differentiation with respect to x with at most one error, as long as $\frac{dy}{dx}$ is eventually correctly linked to $\frac{dx}{dt}$.
- P8** is earned for finding $\frac{dy}{dx}$ and multiplying the result by $\frac{dx}{dt} = 3$.
- P8** can be earned for a stated incorrect $\frac{dy}{dx}$, as long as it is multiplied by 3.
- P9** is only earned for a correct value of $-\frac{2}{3}$ or equivalent.
- Stating $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ alone does not earn any points.

Part B (AB): Graphing calculator not allowed**Question 5****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$.

Model Solution	Scoring
(a) There is a point on the curve near $(2, 4)$ with x -coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y -coordinate of this point.	
$\left. \frac{dy}{dx} \right _{(x, y)=(2, 4)} = \frac{-2(2)}{3 + 4(4)} = -\frac{4}{19}$	Slope of tangent line 1 point
$y \approx 4 - \frac{4}{19}(3 - 2) = \frac{72}{19}$	Approximation 1 point

Scoring notes:

- A response earns the first point for finding $\left. \frac{dy}{dx} \right|_{(x, y)=(2, 4)} = -\frac{4}{19}$, even if this is not labeled or used as the slope of a tangent line.
- A response that does not explicitly find the value of $\left. \frac{dy}{dx} \right|_{(x, y)=(2, 4)}$ but uses a slope of $-\frac{4}{19}$ in a linear approximation also earns the first point.
- A response that declares $\left. \frac{dy}{dx} \right|_{(x, y)=(2, 4)}$ equal to any nonzero value other than $-\frac{4}{19}$ does not earn the first point but is eligible for the second point for a linear approximation at $x = 3$ through the point $(2, 4)$ with a slope of the declared value.
 - The second point cannot be earned with a linear approximation using a slope other than $-\frac{4}{19}$ if that slope has not been declared to be the value of $\left. \frac{dy}{dx} \right|_{(x, y)=(2, 4)}$.
- The second point cannot be earned for an approximation at any value of x other than 3.
- A response does not have to write the tangent line equation but must clearly demonstrate its use at $x = 3$ in finding the requested approximation in order to earn both points.
- The minimal work required to earn both points is $4 - \frac{4}{19}(3 - 2)$.

Total for part (a) 2 points

- (b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

$\frac{dy}{dx} = \frac{-2x}{3+4y} = 0 \Rightarrow x = 0$	Considers $\frac{dy}{dx} = 0$	1 point
And so, if the horizontal line $y = 1$ is tangent to the curve, the point of tangency must be $(0, 1)$. However, the point $(0, 1)$ is not on the curve, because $0^2 + 3 \cdot 1 + 2 \cdot 1^2 = 5 \neq 48$. Therefore, the horizontal line $y = 1$ is not tangent to the curve.	Answer with reason	1 point

Scoring notes:

- The first point can be earned with a response of $\frac{dy}{dx} = 0$, $-2x = 0$, or $x = 0$, OR by identifying the point $(0, 1)$.
- To earn the second point a response must provide a reason that the line $y = 1$ is not tangent to the curve. Merely stating “ $(0, 1)$ does not lie on the curve” is not sufficient to earn the second point.
- Alternate solution:

If $y = 1$, then $x^2 + 3 \cdot 1 + 2 \cdot 1^2 = 48 \Rightarrow x = \pm\sqrt{43}$.

$$\left. \frac{dy}{dx} \right|_{(x,y)=(\pm\sqrt{43},1)} = \frac{\pm 2\sqrt{43}}{7} \neq 0$$

Therefore, the horizontal line $y = 1$ is not tangent to the curve.

- A response that uses this method earns the first point by using $y = 1$ in $x^2 + 3y + 2y^2 = 48$.
- A response that fails to consider both $x = +\sqrt{43}$ and $x = -\sqrt{43}$ does not earn the second point.

Total for part (b) 2 points

- (c) The curve intersects the positive x -axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

At the point $(\sqrt{48}, 0)$, the slope of the line tangent to the curve is $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3 + 4(0)}$.

The denominator of $\frac{dy}{dx}$ is $3 + 4(0)$, which does not equal 0.

Therefore, the line tangent to the curve at this point is not vertical.

Answer with reason

1 point

Scoring notes:

- A response does not need to consider the numerator of $\frac{dy}{dx}\bigg|_{(x,y)=(\sqrt{48},0)}$ in order to earn this point; considering the denominator is sufficient.
- To earn this point a response must clearly demonstrate that the slope of the tangent line at the point $(\sqrt{48}, 0)$ is defined and answer “no.”
 - Such demonstrations include, but are not limited to, the following:
 - $3 + 4(0) \neq 0$
 - At $(\sqrt{48}, 0)$, $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3}$.
 - $\frac{-2\sqrt{48}}{3}$
 - When $3 + 4y = 0$, $y \neq 0$.

Total for part (c)

1 point

- (d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y -coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x -coordinate of the particle's position with respect to time?

$3y^2 \frac{dy}{dt} + 2x \frac{dy}{dt} + 2y \frac{dx}{dt} = 0$	Attempts implicit differentiation	1 point
	$3y^2 \frac{dy}{dt} + 2x \frac{dy}{dt} + 2y \frac{dx}{dt} = 0$	1 point
$\frac{dy}{dt} = -2$ $3(2)^2(-2) + 2(4)(-2) + 2(2) \frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{40}{4} = 10$ The rate of change with respect to time in the x -coordinate is 10 units per second.	Uses $\frac{dy}{dt} = \pm 2$	1 point
	Answer	1 point

Scoring notes:

- The first point is earned for implicitly differentiating $y^3 + 2xy = 24$ with respect to t with at most one error.
 - The first point can also be earned by correctly differentiating with respect to x :

$$3y^2 \frac{dy}{dx} + 2x \frac{dy}{dx} + 2y = 0.$$
- The second point is earned for an equation equivalent to $3y^2 \frac{dy}{dt} + 2x \frac{dy}{dt} + 2y \frac{dx}{dt} = 0$.
- A response does not need to explicitly declare $\frac{dy}{dt} = -2$ or $\frac{dy}{dt} = 2$ in order to earn the third point; this point may be earned by correctly substituting $\frac{dy}{dt} = -2$ or $\frac{dy}{dt} = 2$ in the implicitly differentiated equation. However, a response that uses both $\frac{dy}{dt} = -2$ and $\frac{dy}{dt} = 2$ in the implicitly differentiated equation does not earn the third point.
- The fourth point cannot be earned without the first 3 points. The fourth point is earned only for the value of 10 with no mistakes in supporting work.
 - Note that a response that uses $\frac{dy}{dt} = 2$ and then mishandles subtracting 40 from both sides of the equation, e.g., $3(2)^2(2) + 2(4)(2) + 2(2) \frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{40}{4}$, does not earn the fourth point.

Total for part (d) 4 points**Total for question 5 9 points**

Part B (AB or BC): Graphing calculator not allowed**Question 3****9 points****General Scoring Notes**

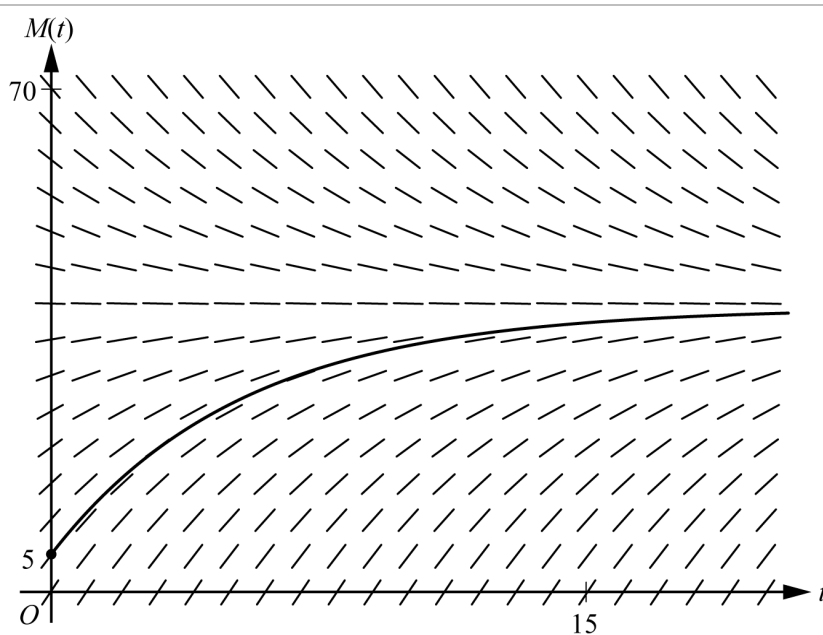
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

Model Solution**Scoring**

- (a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.



Solution curve

1 point**Scoring notes:**

- The solution curve must pass through the point $(0, 5)$, extend reasonably close to the left and right edges of the rectangle, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must lie entirely below the horizontal line segments at $M = 40$.

Total Score (a) 1 point

- (b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

$\left. \frac{dM}{dt} \right _{t=0} = \frac{1}{4}(40 - 5) = \frac{35}{4}$	$\left. \frac{dM}{dt} \right _{t=0}$ 1 point
The tangent line equation is $y = 5 + \frac{35}{4}(t - 0)$. $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$ The temperature of the milk at time $t = 2$ minutes is approximately 22.5° Celsius.	Approximation 1 point

Scoring notes:

- The value of the slope may appear in a tangent line equation or approximation.
- A response of $5 + \frac{35}{4} \cdot 2$ is the minimal response to earn both points.
- A response of $\frac{1}{4}(40 - 5)$ earns the first point. If there are any subsequent errors in simplification, the response does not earn the second point.
- In order to earn the second point the response must present an approximation found by using a tangent line that:
 - passes through the point $(0, 5)$ and
 - has slope $\frac{35}{4}$ or a nonzero slope that is declared to be the value of $\frac{dM}{dt}$.
- An unsupported approximation does not earn the second point.
- The approximation need not be simplified, but the response does not earn the second point if the approximation is simplified incorrectly.

Total for part (b) 2 points

- (c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M . Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

$\frac{d^2M}{dt^2} = -\frac{1}{4} \frac{dM}{dt} = -\frac{1}{4} \left(\frac{1}{4}(40 - M) \right) = -\frac{1}{16}(40 - M)$	$\frac{d^2M}{dt^2}$ 1 point
Because $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down. Therefore, the tangent line approximation of $M(2)$ is an overestimate.	Overestimate with reason 1 point

Scoring notes:

- The first point is earned for either $\frac{d^2M}{dt^2} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right)$ or $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$ (or equivalent). A response that presents any subsequent simplification error does not earn the second point.
- A response that presents an expression for $\frac{d^2M}{dt^2}$ in terms of $\frac{dM}{dt}$ but fails to continue to an expression in terms of M (i.e., $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt}$) does not earn the first point but is eligible for the second point.
- If the response presents an expression for $\frac{d^2M}{dt^2}$ that is incorrect, the response is eligible for the second point only if the expression is a nonconstant linear function that is negative for $5 < M < 40$.
 - Special case: A response that presents $\frac{d^2M}{dt^2} = \frac{1}{16}(40 - M)$ does not earn the first point but is eligible to earn the second point for a consistent answer and reason.
- To earn the second point a response must include $\frac{d^2M}{dt^2} < 0$, or $\frac{dM}{dt}$ is decreasing, or the graph of M is concave down, as well as the conclusion that the approximation is an overestimate.
- A response that presents an argument based on $\frac{d^2M}{dt^2}$ or concavity at a single point does not earn the second point.

Total for part (c) 2 points

- (d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

$\frac{dM}{40 - M} = \frac{1}{4} dt$ $\int \frac{dM}{40 - M} = \int \frac{1}{4} dt$	Separates variables	1 point
$-\ln 40 - M = \frac{1}{4}t + C$	Finds antiderivatives	1 point
$-\ln 40 - 5 = 0 + C \Rightarrow C = -\ln 35$ $M(t) < 40 \Rightarrow 40 - M > 0 \Rightarrow 40 - M = 40 - M$ $-\ln(40 - M) = \frac{1}{4}t - \ln 35$ $\ln(40 - M) = -\frac{1}{4}t + \ln 35$	Constant of integration and uses initial condition	1 point

$$40 - M = 35e^{-t/4}$$

$$M = 40 - 35e^{-t/4}$$

Solves for M **1 point****Scoring notes:**

- A response with no separation of variables earns 0 out of 4 points.
- A response that presents an antiderivative of $-\ln(40 - M)$ without absolute value symbols is eligible for all 4 points.
- A response with no constant of integration can earn at most the first 2 points.
- A response is eligible for the third point only if it has earned the first 2 points.
 - Special Case: A response that presents $+\ln(40 - M) = \frac{t}{4} + C$ (or equivalent) does not earn the second point, is eligible for the third point, but not eligible for the fourth.
- An eligible response earns the third point by correctly including the constant of integration in an equation and substituting 0 for t and 5 for M .
- A response is eligible for the fourth point only if it has earned the first 3 points.
- A response earns the fourth point only for an answer of $M = 40 - 35e^{-t/4}$ or equivalent.

Total for part (d) 4 points**Total for question 3 9 points**

Part B (AB): Graphing calculator not allowed**Question 5****9 points****General Scoring Notes**

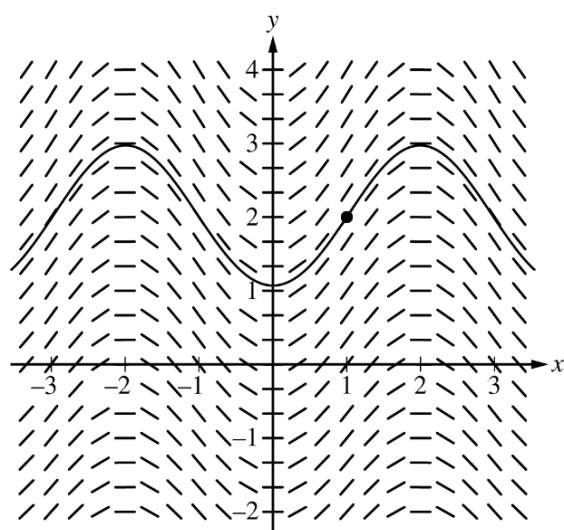
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

Model Solution**Scoring**

- (a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.



Solution curve

1 point**Scoring notes:**

- The solution curve must pass through the point $(1, 2)$, extend reasonably close to the left and right edges of the square and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must indicate $f(x) > 0$ for all points on the curve.
- All local maximum/minimum points on the solution curve must occur at horizontal line segments in the slope field.

Total for part (a)**1 point**

- (b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

$\left. \frac{dy}{dx} \right _{(x,y)=(1,2)} = \frac{1}{2} \cdot 3 \cdot \sin\left(\frac{\pi}{2}\right) = \frac{3}{2}$ An equation for the tangent line is $y = 2 + \frac{3}{2}(x - 1)$.	Tangent line equation	1 point
$f(0.8) \approx 2 + \frac{3}{2}(0.8 - 1) = 1.7$	Approximation	1 point

Scoring notes:

- The tangent line equation can be presented in any equivalent form.
- An incorrect tangent line equation with a slope of $\frac{3}{2}$ is eligible to earn the second point for a consistent answer.
- A response of only $2 + \frac{3}{2}(0.8 - 1)$ earns the second point but not the first.

Total for part (b) 2 points

- (c) It is known that $f''(x) > 0$ for $-1 \leq x \leq 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Because $f''(x) > 0$, f is concave up on $-1 \leq x \leq 1$, the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ is an underestimate.	Answer with reason	1 point
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Scoring notes:

- The reason must include $f''(x) > 0$, $f'(x)$ is increasing, or $f(x)$ is concave up.

Total for part (c) 1 point

- (d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) \sqrt{y+7} \text{ with the initial condition } f(1) = 2.$$

$\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) dx$	Separation of variables	1 point
$2\sqrt{y+7} = -\frac{1}{\pi} \cos\left(\frac{\pi}{2}x\right) + C$	One correct antiderivative	1 point
	The other correct antiderivative	1 point
$f(1) = 2 \Rightarrow 2\sqrt{2+7} = -\frac{1}{\pi} \cos\left(\frac{\pi}{2} \cdot 1\right) + C$ $\Rightarrow 6 = -\frac{1}{\pi} \cos\left(\frac{\pi}{2}\right) + C \Rightarrow C = 6$ $\sqrt{y+7} = 3 - \frac{1}{2\pi} \cos\left(\frac{\pi}{2}x\right)$	Constant of integration and uses initial condition	1 point
$y = \left(3 - \frac{1}{2\pi} \cos\left(\frac{\pi}{2}x\right)\right)^2 - 7$	Solves for y	1 point

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points.
- A response with no constant of integration can earn at most the first 3 points.
- A response is eligible for the fourth point only if it has earned the first point and at least 1 of the 2 antiderivative points.
 - Special case: The incorrect separation of $\sqrt{y+7} \, dy = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) dx$ does not earn the first point, is only eligible for the antiderivative point for $-\frac{1}{\pi} \cos\left(\frac{\pi}{2}x\right)$, and is eligible for the fourth point.
- An eligible response earns the fourth point by correctly including the constant of integration in an equation and substituting 1 for x and 2 for y .
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (d) 5 points

Total for question 5 9 points

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Scoring Guidelines

2018 SCORING GUIDELINES

Question 1

(a) $\int_0^{300} r(t) dt = 270$

According to the model, 270 people enter the line for the escalator during the time interval $0 \leq t \leq 300$.

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $20 + \int_0^{300} (r(t) - 0.7) dt = 20 + \int_0^{300} r(t) dt - 0.7 \cdot 300 = 80$

According to the model, 80 people are in line at time $t = 300$.

2 : $\begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$

(c) Based on part (b), the number of people in line at time $t = 300$ is 80.

The first time t that there are no people in line is

$$300 + \frac{80}{0.7} = 414.286 \text{ (or 414.285) seconds.}$$

1 : answer

(d) The total number of people in line at time t , $0 \leq t \leq 300$, is modeled by

$$20 + \int_0^t r(x) dx - 0.7t.$$

$$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298, t_2 = 166.574719$$

4 : $\begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

t	People in line for escalator
0	20
t_1	3.803
t_2	158.070
300	80

The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line.

2018 SCORING GUIDELINES

Question 2

(a) $v'(3) = -2.118$

The acceleration of the particle at time $t = 3$ is -2.118 .

1 : answer

(b) $x(3) = x(0) + \int_0^3 v(t) dt = -5 + \int_0^3 v(t) dt = -1.760213$

The position of the particle at time $t = 3$ is -1.760 .

3 : $\begin{cases} 1 : \int_0^3 v(t) dt \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^{3.5} v(t) dt = 2.844$ (or 2.843)

$$\int_0^{3.5} |v(t)| dt = 3.737$$

The integral $\int_0^{3.5} v(t) dt$ is the displacement of the particle over the time interval $0 \leq t \leq 3.5$.

The integral $\int_0^{3.5} |v(t)| dt$ is the total distance traveled by the particle over the time interval $0 \leq t \leq 3.5$.

3 : $\begin{cases} 1 : \text{answers} \\ 2 : \text{interpretations of } \int_0^{3.5} v(t) dt \\ \text{and } \int_0^{3.5} |v(t)| dt \end{cases}$

(d) $v(t) = x_2'(t)$

$$v(t) = 2t - 1 \Rightarrow t = 1.57054$$

The two particles are moving with the same velocity at time $t = 1.571$ (or 1.570).

2 : $\begin{cases} 1 : \text{sets } v(t) = x_2'(t) \\ 1 : \text{answer} \end{cases}$

2018 SCORING GUIDELINES

Question 3

$$\begin{aligned} \text{(a)} \quad f(-5) &= f(1) + \int_1^{-5} g(x) \, dx = f(1) - \int_{-5}^1 g(x) \, dx \\ &= 3 - \left(-9 - \frac{3}{2} + 1 \right) = 3 - \left(-\frac{19}{2} \right) = \frac{25}{2} \end{aligned}$$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b)} \quad \int_1^6 g(x) \, dx &= \int_1^3 g(x) \, dx + \int_3^6 g(x) \, dx \\ &= \int_1^3 2 \, dx + \int_3^6 2(x-4)^2 \, dx \\ &= 4 + \left[\frac{2}{3}(x-4)^3 \right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3} \right) = 10 \end{aligned}$$

3 : $\begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$

(c) The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f''(x) = g'(x)$ is increasing on those intervals.

2 : $\begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(d) The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

2018 SCORING GUIDELINES

Question 4

$$(a) \quad H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$$(b) \quad \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because H is differentiable on $3 \leq t \leq 5$, H is continuous on $3 \leq t \leq 5$.

By the Mean Value Theorem, there exists a value c , $3 < c < 5$, such that $H'(c) = 2$.

$$(c) \quad \text{The average height of the tree over the time interval } 2 \leq t \leq 10 \text{ is given by } \frac{1}{10 - 2} \int_2^{10} H(t) \, dt.$$

$$\begin{aligned} \frac{1}{8} \int_2^{10} H(t) \, dt &\approx \frac{1}{8} \left(\frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \right) \\ &= \frac{1}{8} (65.75) = \frac{263}{32} \end{aligned}$$

The average height of the tree over the time interval $2 \leq t \leq 10$ is $\frac{263}{32}$ meters.

$$(d) \quad G(x) = 50 \Rightarrow x = 1$$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\frac{3}{4}$ meter per year.

$$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$$

$$2 : \begin{cases} 1 : \frac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$$

$$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$$

$$3 : \begin{cases} 2 : \frac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$$

Note: max 1/3 [1-0] if no chain rule

2018 SCORING GUIDELINES

Question 5

- (a) The average rate of change of
- f
- on the interval
- $0 \leq x \leq \pi$
- is

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{-e^\pi - 1}{\pi}.$$

1 : answer

- (b)
- $f'(x) = e^x \cos x - e^x \sin x$

$$f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$$

2 : $\begin{cases} 1 : f'(x) \\ 1 : \text{slope} \end{cases}$

The slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$ is $e^{3\pi/2}$.

- (c)
- $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

x	$f(x)$
0	1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}e^{\pi/4}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}e^{5\pi/4}$
2π	$e^{2\pi}$

3 : $\begin{cases} 1 : \text{sets } f'(x) = 0 \\ 1 : \text{identifies } x = \frac{\pi}{4}, x = \frac{5\pi}{4} \\ \text{as candidates} \\ 1 : \text{answer with justification} \end{cases}$

The absolute minimum value of f on $0 \leq x \leq 2\pi$ is $-\frac{1}{\sqrt{2}}e^{5\pi/4}$.

- (d)
- $\lim_{x \rightarrow \pi/2} f(x) = 0$

Because g is differentiable, g is continuous.

$$\lim_{x \rightarrow \pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$$

By L'Hospital's Rule,

$$\lim_{x \rightarrow \pi/2} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \pi/2} \frac{f'(x)}{g'(x)} = \frac{-e^{\pi/2}}{2}.$$

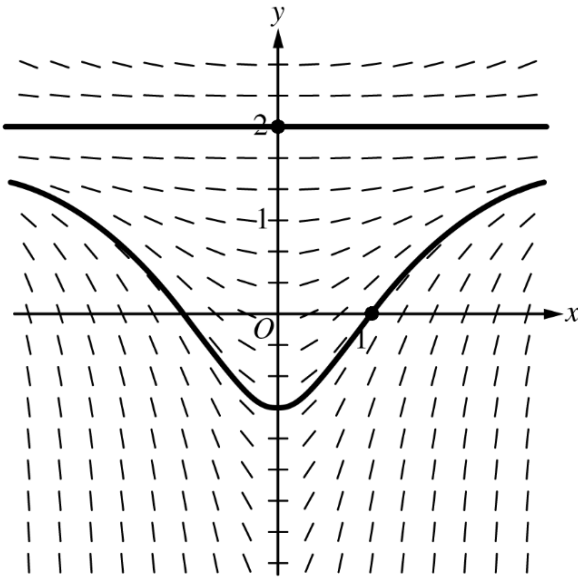
3 : $\begin{cases} 1 : g \text{ is continuous at } x = \frac{\pi}{2} \\ \text{and limits equal 0} \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

Note: max 1/3 [1-0-0] if no limit notation attached to a ratio of derivatives

2018 SCORING GUIDELINES

Question 6

(a)



$$2 : \begin{cases} 1 : \text{solution curve through } (0, 2) \\ 1 : \text{solution curve through } (1, 0) \end{cases}$$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

$$(b) \left. \frac{dy}{dx} \right|_{(x, y)=(1, 0)} = \frac{4}{3}$$

An equation for the line tangent to the graph of $y = f(x)$ at $x = 1$ is $y = \frac{4}{3}(x - 1)$.

$$f(0.7) \approx \frac{4}{3}(0.7 - 1) = -0.4$$

$$2 : \begin{cases} 1 : \text{equation of tangent line} \\ 1 : \text{approximation} \end{cases}$$

$$(c) \frac{dy}{dx} = \frac{1}{3}x(y - 2)^2$$

$$\int \frac{dy}{(y - 2)^2} = \int \frac{1}{3}x \, dx$$

$$\frac{-1}{y - 2} = \frac{1}{6}x^2 + C$$

$$\frac{1}{2} = \frac{1}{6} + C \Rightarrow C = \frac{1}{3}$$

$$\frac{-1}{y - 2} = \frac{1}{6}x^2 + \frac{1}{3} = \frac{x^2 + 2}{6}$$

$$y = 2 - \frac{6}{x^2 + 2}$$

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \text{and uses initial condition} \\ 1 : \text{solves for } y \end{cases}$$

Note: 0/5 if no separation of variables

Note: max 3/5 [1-2-0-0] if no constant of integration

Note: this solution is valid for $-\infty < x < \infty$.

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2016 SCORING GUIDELINES

Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \leq t \leq 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \leq t \leq 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

- (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.
- (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.
- (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.
- (d) For $0 \leq t \leq 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

$$(a) \quad R'(2) \approx \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{3 - 1} = -120 \text{ liters/hr}^2$$

$$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$$

(b) The total amount of water removed is given by $\int_0^8 R(t) dt$.

$$\begin{aligned} \int_0^8 R(t) dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

This is an overestimate since R is a decreasing function.

$$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$$

$$\begin{aligned} (c) \quad \text{Total} &\approx 50000 + \int_0^8 W(t) dt - 8050 \\ &= 50000 + 7836.195325 - 8050 \approx 49786 \text{ liters} \end{aligned}$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$$

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

$$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$$

Therefore, the Intermediate Value Theorem guarantees at least one time t , $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t , the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

2016 SCORING GUIDELINES

Question 2

For $t \geq 0$, a particle moves along the x -axis. The velocity of the particle at time t is given by

$v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right)$. The particle is at position $x = 2$ at time $t = 4$.

- At time $t = 4$, is the particle speeding up or slowing down?
- Find all times t in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.
- Find the position of the particle at time $t = 0$.
- Find the total distance the particle travels from time $t = 0$ to time $t = 3$.

(a) $v(4) = 2.978716 > 0$
 $v'(4) = -1.164000 < 0$

The particle is slowing down since the velocity and acceleration have different signs.

(b) $v(t) = 0 \Rightarrow t = 2.707468$

$v(t)$ changes from positive to negative at $t = 2.707$.
 Therefore, the particle changes direction at this time.

(c) $x(0) = x(4) + \int_4^0 v(t) dt$
 $= 2 + (-5.815027) = -3.815$

(d) Distance $= \int_0^3 |v(t)| dt = 5.301$

2 : conclusion with reason

2 : $\begin{cases} 1 : t = 2.707 \\ 1 : \text{justification} \end{cases}$

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

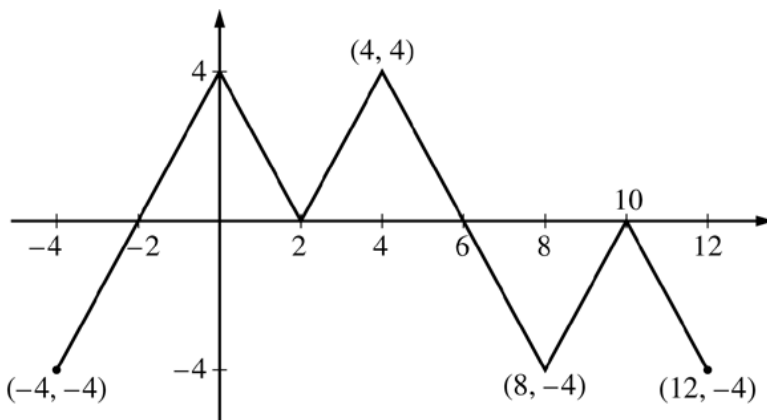
2016 SCORING GUIDELINES

Question 3

The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$, the function g is defined by

$$g(x) = \int_2^x f(t) \, dt.$$

- (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
- (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
- (d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

Graph of f

- (a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \leq 0$ for $8 \leq x \leq 12$.
- (b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \leq x \leq 4$ and decreasing for $4 \leq x \leq 8$.
- (c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \leq x \leq 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

- (d) $g(x) \leq 0$ for $-4 \leq x \leq 2$ and $10 \leq x \leq 12$.

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

1 : answer with justification

1 : answer with justification

4 : $\left\{ \begin{array}{l} 1 : \text{considers } x = -2 \text{ and } x = 6 \\ \quad \text{as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{array} \right.$

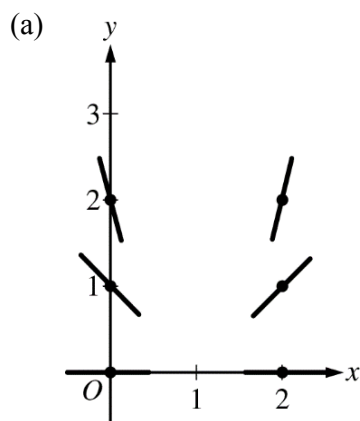
2 : intervals

2016 SCORING GUIDELINES

Question 4

Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$.
Use your equation to approximate $f(2.1)$.
- (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.



2 : $\begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = \frac{3^2}{2-1} = 9$

An equation for the tangent line is $y = 9(x - 2) + 3$.

$$f(2.1) \approx 9(2.1 - 2) + 3 = 3.9$$

2 : $\begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(c) $\frac{1}{y^2} dy = \frac{1}{x-1} dx$
 $\int \frac{1}{y^2} dy = \int \frac{1}{x-1} dx$
 $-\frac{1}{y} = \ln|x-1| + C$
 $-\frac{1}{3} = \ln|2-1| + C \Rightarrow C = -\frac{1}{3}$
 $-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$
 $y = \frac{1}{\frac{1}{3} - \ln(x-1)}$

5 : $\begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

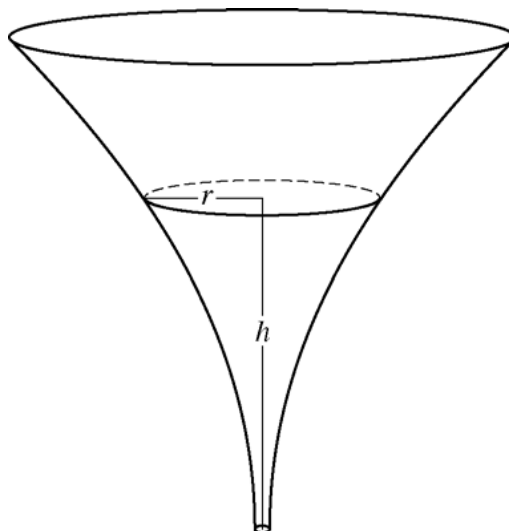
Note: max 3/5 [1-2-0-0] if no constant of integration

Note: 0/5 if no separation of variables

Note: This solution is valid for $1 < x < 1 + e^{1/3}$.

2016 SCORING GUIDELINES

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h , the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$, where $0 \leq h \leq 10$. The units of r and h are inches.

- (a) Find the average value of the radius of the funnel.
- (b) Find the volume of the funnel.
- (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$\begin{aligned} \text{(a) Average radius} &= \frac{1}{10} \int_0^{10} \frac{1}{20}(3 + h^2) dh = \frac{1}{200} \left[3h + \frac{h^3}{3} \right]_0^{10} \\ &= \frac{1}{200} \left(\left(30 + \frac{1000}{3} \right) - 0 \right) = \frac{109}{60} \text{ in} \end{aligned}$$

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b) Volume} &= \pi \int_0^{10} \left(\left(\frac{1}{20} \right) (3 + h^2) \right)^2 dh = \frac{\pi}{400} \int_0^{10} (9 + 6h^2 + h^4) dh \\ &= \frac{\pi}{400} \left[9h + 2h^3 + \frac{h^5}{5} \right]_0^{10} \\ &= \frac{\pi}{400} \left(\left(90 + 2000 + \frac{100000}{5} \right) - 0 \right) = \frac{2209\pi}{40} \text{ in}^3 \end{aligned}$$

3 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(c) } \frac{dr}{dt} &= \frac{1}{20}(2h) \frac{dh}{dt} \\ -\frac{1}{5} &= \frac{3}{10} \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3} \text{ in/sec} \end{aligned}$$

3 : $\begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

2016 SCORING GUIDELINES

Question 6

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-6	3	2	8
2	2	-2	-3	0
3	8	7	6	2
6	4	5	3	-1

The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x .

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

(b) Let $h(x) = \frac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\int_1^3 f''(2x) dx$.

$$\begin{aligned} \text{(a)} \quad k(3) &= f(g(3)) = f(6) = 4 \\ k'(3) &= f'(g(3)) \cdot g'(3) = f'(6) \cdot 2 = 5 \cdot 2 = 10 \end{aligned}$$

An equation for the tangent line is $y = 10(x - 3) + 4$.

$$\begin{aligned} \text{(b)} \quad h'(1) &= \frac{f(1) \cdot g'(1) - g(1) \cdot f'(1)}{(f(1))^2} \\ &= \frac{(-6) \cdot 8 - 2 \cdot 3}{(-6)^2} = \frac{-54}{36} = -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_1^3 f''(2x) dx &= \frac{1}{2} [f'(2x)]_1^3 = \frac{1}{2} [f'(6) - f'(2)] \\ &= \frac{1}{2} [5 - (-2)] = \frac{7}{2} \end{aligned}$$

3 : $\begin{cases} 2 : \text{slope at } x = 3 \\ 1 : \text{equation for tangent line} \end{cases}$

3 : $\begin{cases} 2 : \text{expression for } h'(1) \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

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2014 SCORING GUIDELINES

Question 1

Grass clippings are placed in a bin, where they decompose. For $0 \leq t \leq 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

- Find the average rate of change of $A(t)$ over the interval $0 \leq t \leq 30$. Indicate units of measure.
- Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.
- Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \leq t \leq 30$.
- For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

$$(a) \frac{A(30) - A(0)}{30 - 0} = -0.197 \text{ (or } -0.196) \text{ lbs/day}$$

1 : answer with units

$$(b) A'(15) = -0.164 \text{ (or } -0.163)$$

The amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lbs/day at time $t = 15$ days.

2 : $\begin{cases} 1 : A'(15) \\ 1 : \text{interpretation} \end{cases}$

$$(c) A(t) = \frac{1}{30} \int_0^{30} A(t) dt \Rightarrow t = 12.415 \text{ (or } 12.414)$$

2 : $\begin{cases} 1 : \frac{1}{30} \int_0^{30} A(t) dt \\ 1 : \text{answer} \end{cases}$

$$(d) L(t) = A(30) + A'(30) \cdot (t - 30)$$

$$A'(30) = -0.055976$$

$$A(30) = 0.782928$$

$$L(t) = 0.5 \Rightarrow t = 35.054$$

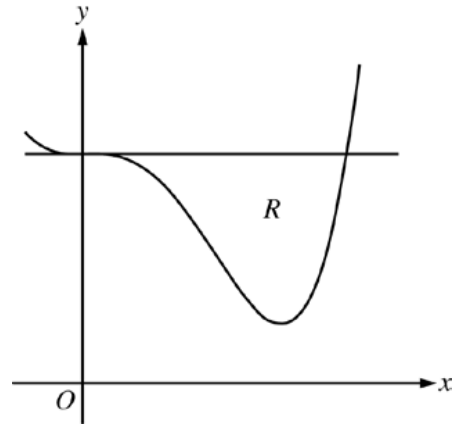
4 : $\begin{cases} 2 : \text{expression for } L(t) \\ 1 : L(t) = 0.5 \\ 1 : \text{answer} \end{cases}$

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Question 2

Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

- (a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.
- (b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x -axis is an isosceles right triangle with a leg in R . Find the volume of the solid.
- (c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k .



(a) $f(x) = 4 \Rightarrow x = 0, 2.3$

$$\begin{aligned} \text{Volume} &= \pi \int_0^{2.3} \left[(4 + 2)^2 - (f(x) + 2)^2 \right] dx \\ &= 98.868 \text{ (or } 98.867) \end{aligned}$$

$$4 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$$

(b) $\text{Volume} = \int_0^{2.3} \frac{1}{2} (4 - f(x))^2 dx$
 $= 3.574 \text{ (or } 3.573)$

$$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(c) $\int_0^k (4 - f(x)) dx = \int_k^{2.3} (4 - f(x)) dx$

$$2 : \begin{cases} 1 : \text{area of one region} \\ 1 : \text{equation} \end{cases}$$

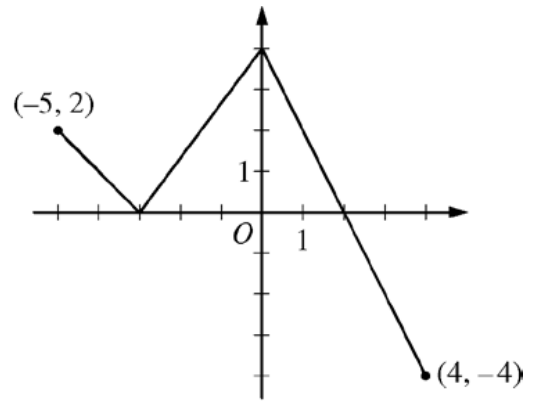
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Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

Let g be the function defined by $g(x) = \int_{-3}^x f(t) dt$.

- (a) Find $g(3)$.
- (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.
- (c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.
- (d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

Graph of f

(a) $g(3) = \int_{-3}^3 f(t) dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \frac{5xg'(x) - g(x)5}{(5x)^2} = \frac{5xg'(x) - 5g(x)}{25x^2}$

3 : $\begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} h'(3) &= \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2} \\ &= \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3} \end{aligned}$$

(d) $p'(x) = f'(x^2 - x)(2x - 1)$

3 : $\begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

$$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$$

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Question 4

Train A runs back and forth on an east-west section of railroad track. Train A 's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

- (a) Find the average acceleration of train A over the interval $2 \leq t \leq 8$.
- (b) Do the data in the table support the conclusion that train A 's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.
- (c) At time $t = 2$, train A 's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A , in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.
- (d) A second train, train B , travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

(a) average accel = $\frac{v_A(8) - v_A(2)}{8 - 2} = \frac{-120 - 100}{6} = -\frac{110}{3} \text{ m/min}^2$

(b) v_A is differentiable $\Rightarrow v_A$ is continuous
 $v_A(8) = -120 < -100 < 40 = v_A(5)$

Therefore, by the Intermediate Value Theorem, there is a time t , $5 < t < 8$, such that $v_A(t) = -100$.

(c) $s_A(12) = s_A(2) + \int_2^{12} v_A(t) dt = 300 + \int_2^{12} v_A(t) dt$
 $\int_2^{12} v_A(t) dt \approx 3 \cdot \frac{100 + 40}{2} + 3 \cdot \frac{40 - 120}{2} + 4 \cdot \frac{-120 - 150}{2}$
 $= -450$

$$s_A(12) \approx 300 - 450 = -150$$

The position of Train A at time $t = 12$ minutes is approximately 150 meters west of Origin Station.

- (d) Let x be train A 's position, y train B 's position, and z the distance between train A and train B .

$$z^2 = x^2 + y^2 \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$x = 300, y = 400 \Rightarrow z = 500$$

$$v_B(2) = -20 + 120 + 25 = 125$$

$$500 \frac{dz}{dt} = (300)(100) + (400)(125)$$

$$\frac{dz}{dt} = \frac{80000}{500} = 160 \text{ meters per minute}$$

1 : average acceleration

2 : $\begin{cases} 1 : v_A(8) < -100 < v_A(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$

3 : $\begin{cases} 1 : \text{position expression} \\ 1 : \text{trapezoidal sum} \\ 1 : \text{position at time } t = 12 \end{cases}$

3 : $\begin{cases} 2 : \text{implicit differentiation of} \\ \text{distance relationship} \\ 1 : \text{answer} \end{cases}$

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Question 5

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

The twice-differentiable functions f and g are defined for all real numbers x . Values of f , f' , g , and g' for various values of x are given in the table above.

- (a) Find the x -coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.
- (b) Explain why there must be a value c , for $-1 < c < 1$, such that $f''(c) = 0$.
- (c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.
- (d) Evaluate $\int_{-2}^3 f'(g(x))g'(x) dx$.

- (a) $x = 1$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a relative minimum at $x = 1$.

- (b) f' is differentiable $\Rightarrow f'$ is continuous on the interval $-1 \leq x \leq 1$

$$\frac{f'(1) - f'(-1)}{1 - (-1)} = \frac{0 - 0}{2} = 0$$

Therefore, by the Mean Value Theorem, there is at least one value c , $-1 < c < 1$, such that $f''(c) = 0$.

- (c) $h'(x) = \frac{1}{f(x)} \cdot f'(x)$

$$h'(3) = \frac{1}{f(3)} \cdot f'(3) = \frac{1}{7} \cdot \frac{1}{2} = \frac{1}{14}$$

- (d) $\int_{-2}^3 f'(g(x))g'(x) dx = \left[f(g(x)) \right]_{x=-2}^{x=3}$
 $= f(g(3)) - f(g(-2))$
 $= f(1) - f(-1)$
 $= 2 - 8 = -6$

1 : answer with justification

$$2 : \begin{cases} 1 : f'(1) - f'(-1) = 0 \\ 1 : \text{explanation, using Mean Value Theorem} \end{cases}$$

$$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 2 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$$

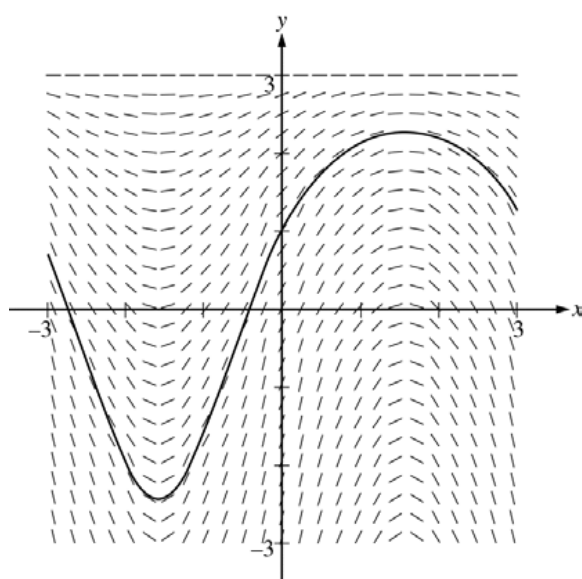
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Question 6

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

- (a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.
- (b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.
- (c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

(a) 1 : solution curve



(b) $\left. \frac{dy}{dx} \right|_{(x,y)=(0,1)} = 2 \cos 0 = 2$

An equation for the tangent line is $y = 2x + 1$.

$$f(0.2) \approx 2(0.2) + 1 = 1.4$$

2 : $\begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(c) $\frac{dy}{dx} = (3 - y)\cos x$

$$\int \frac{dy}{3 - y} = \int \cos x \, dx$$

$$-\ln|3 - y| = \sin x + C$$

$$-\ln 2 = \sin 0 + C \Rightarrow C = -\ln 2$$

$$-\ln|3 - y| = \sin x - \ln 2$$

Because $y(0) = 1$, $y < 3$, so $|3 - y| = 3 - y$

$$3 - y = 2e^{-\sin x}$$

$$y = 3 - 2e^{-\sin x}$$

6 : $\begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables