## Part B (AB): Graphing calculator not allowed

## Question 6 — 9 points

### General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve $G$ defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

| | Model Solution | Scoring | |
|---|---|---|---|
| **A** | Show that $\frac{dy}{dx} = \frac{-x}{2\left(3y^2 - 2y - 1\right)}$. | | |
| | $\frac{d}{dx}\left(y^3 - y^2 - y + \frac{1}{4}x^2\right) = \frac{d}{dx}(0)$ <br> $\Rightarrow 3y^2 \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} - \frac{dy}{dx} + \frac{x}{2} = 0$ | Implicit differentiation | **Point 1 (P1)** |
| | $\Rightarrow \left(3y^2 - 2y - 1\right)\frac{dy}{dx} = -\frac{x}{2}$ <br> $\Rightarrow \frac{dy}{dx} = \frac{-x}{2\left(3y^2 - 2y - 1\right)}$ | Verification | **Point 2 (P2)** |

### Scoring Notes for Part A

- **P1** is earned only for the correct implicit differentiation of $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$. Responses may use alternative notations for $\frac{dy}{dx}$, such as $y'$.
- To be eligible for **P2**, a response must have earned **P1**.
- It is sufficient to present $\left(3y^2 - 2y - 1\right)\frac{dy}{dx} = -\frac{x}{2}$ to earn **P2**, provided there are no subsequent errors.





| | Model Solution | Scoring | |
|---|---|---|---|
| **B** | There is a point $P$ on the curve $G$ near $(2, -1)$ with $x$-coordinate 1.6. Use the line tangent to the curve at $(2, -1)$ to approximate the $y$-coordinate of point $P$. | | |
| | $\left.\frac{dy}{dx}\right\|_{(x,\,y)=(2,\,-1)} = \frac{-2}{2(3+2-1)} = -\frac{1}{4}$ | Slope of tangent line | **Point 3 (P3)** |
| | $y \approx -1 - \frac{1}{4}(1.6 - 2) = -0.9$ | Tangent line approximation | **Point 4 (P4)** |

### Scoring Notes for Part B

- A response can earn **P3** with $\left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,-1)} = -\frac{1}{4}$, $\frac{dy}{dx} = -\frac{1}{4}$, "slope is $-\frac{1}{4}$," or equivalent.
- A response that presents a linear approximation with a slope of $-\frac{1}{4}$ also earns **P3**.
- A response that declares $\left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,-1)}$ (or the slope) equal to any nonzero value $k \neq -\frac{1}{4}$ does not earn **P3**. Such a response earns **P4** for a presented approximation mathematically equivalent to $-1 + k(-0.4)$.
- **P4** cannot be earned with a linear approximation using a slope other than $-\frac{1}{4}$ if that slope has not been declared to be the value of $\left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,-1)}$.
- A response does not have to present the tangent line equation but must clearly demonstrate its use at $x = 1.6$ in finding the requested approximation to be eligible for **P4**.
- A response of $-1 - \frac{1}{4}(-0.4)$ earns both **P3** and **P4**.
- A response of $-1 - \frac{1}{4}(1.6 - 2)$ or equivalent banks **P4** (i.e., subsequent errors in simplification will not be considered in scoring for **P4)**.

  Note: An ambiguous response, such as $-1 - \frac{1}{4}(1.6 - 2$, does not bank **P4** and therefore must go on to resolve the ambiguity with a correct final answer (e.g., $-0.9$) to earn **P4**.

| | | | |
|---|---|---|---|
| **C** | For $x > 0$ and $y > 0$, there is a point $S$ on the curve $G$ at which the line tangent to the curve at that point is vertical. Find the $y$-coordinate of point $S$. Show the work that leads to your answer. | | |
| | For $x > 0$, the curve $G$ has a vertical tangent line when $2\left(3y^2 - 2y - 1\right) = 0$. | Sets denominator equal to 0 | **Point 5 (P5)** |
| | $2\left(3y^2 - 2y - 1\right) = 0 \Rightarrow 2(3y+1)(y-1) = 0$<br>Because $y > 0$, it follows that $y = 1$.<br><br>The line tangent to the curve is vertical at the point on the curve where $y = 1$. | Answer | **Point 6 (P6)** |

**Scoring Notes for Part C**

- **P5** is earned with any of $2\left(3y^2 - 2y - 1\right) = 0$, $3y^2 - 2y - 1 = 0$, $2(3y \pm 1)(y \pm 1) = 0$, or $(3y \pm 1)(y \pm 1) = 0$.
- To be eligible for **P6**, a response must have earned **P5**.
- A response does not need to consider the numerator of $\frac{dy}{dx} = \frac{-x}{2\left(3y^2 - 2y - 1\right)}$ to earn **P5** or **P6**; considering the denominator is sufficient.
- A response that states solutions of $y = -\frac{1}{3}$ and $y = 1$, but does not clearly identify that $y = 1$ is the only solution to the prompt, does not earn **P6**.
- In the presence of algebraic work to find the value of $y$, **P6** is earned only if the algebraic work is correct.
- A response of "$2\left(3y^2 - 2y - 1\right) = 0$, $y = 1$" earns both **P5** and **P6**.





| | | | |
|---|---|---|---|
| **D** | A particle moves along the curve $H$ defined by the equation $2xy + \ln y = 8$. At the instant when the particle is at the point $(4, 1)$, $\frac{dx}{dt} = 3$. Find $\frac{dy}{dt}$ at that instant. Show the work that leads to your answer. | | |
| | $\frac{d}{dt}(2xy + \ln y) = \frac{d}{dt}(8)$<br><br>$2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$ | Attempts implicit differentiation with respect to $t$ | **Point 7 (P7)** |
| | | $2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$ | **Point 8 (P8)** |
| | $2(3)(1) + 2(4)\frac{dy}{dt} + \frac{1}{1}\frac{dy}{dt} = 0$<br>$\Rightarrow 6 + 9\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{2}{3}$ | Answer | **Point 9 (P9)** |

**Scoring Notes for Part D**

- **P7** is earned for implicitly differentiating $2xy + \ln y = 8$ with respect to $t$ with at most one error.
- **P8** is earned for an equation equivalent to $2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$.
- To be eligible for **P9**, a response must have earned **P7** and **P8**, with no errors in implicit differentiation.
- **P9** is earned only for the value of $-\frac{2}{3}$.
- Alternate solution:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{d}{dx}(2xy + \ln y) = \frac{d}{dx}(8) \Rightarrow 2y + 2x\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2y}{2x + \frac{1}{y}}$$

$$\left.\frac{dy}{dx}\right|_{(x,\, y)=(4,\, 1)} = \frac{-2(1)}{2(4) + \frac{1}{1}} = -\frac{2}{9}$$

$$\left.\frac{dy}{dt}\right|_{(x,\, y)=(4,\, 1)} = \left.\frac{dy}{dx} \cdot \frac{dx}{dt}\right|_{(x,\, y)=(4,\, 1)} = -\frac{2}{9} \cdot 3 = -\frac{2}{3}$$

  - **P7** is earned for an implicit differentiation with respect to $x$ with at most one error, as long as $\frac{dy}{dx}$ is eventually correctly linked to $\frac{dx}{dt}$.
  - **P8** is earned for finding $\frac{dy}{dx}$ and multiplying the result by $\frac{dx}{dt} = 3$.
  - **P8** can be earned for a stated incorrect $\frac{dy}{dx}$, as long as it is multiplied by 3.
  - **P9** is only earned for a correct value of $-\frac{2}{3}$ or equivalent.
  - Stating $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ alone does not earn any points.

## Part B (AB): Graphing calculator not allowed
## Question 5 — 9 points

**General Scoring Notes**
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3+4y}$.

| Model Solution | Scoring | |
|---|---|---|

**(a)** There is a point on the curve near $(2, 4)$ with $x$-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the $y$-coordinate of this point.

| | | |
|---|---|---|
| $\left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(2,\,4)} = \frac{-2(2)}{3+4(4)} = -\frac{4}{19}$ | Slope of tangent line | **1 point** |
| $y \approx 4 - \frac{4}{19}(3-2) = \frac{72}{19}$ | Approximation | **1 point** |

**Scoring notes:**

- A response earns the first point for finding $\left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(2,\,4)} = -\frac{4}{19}$, even if this is not labeled or used as the slope of a tangent line.
- A response that does not explicitly find the value of $\left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(2,\,4)}$ but uses a slope of $-\frac{4}{19}$ in a linear approximation also earns the first point.
- A response that declares $\left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(2,\,4)}$ equal to any nonzero value other than $-\frac{4}{19}$ does not earn the first point but is eligible for the second point for a linear approximation at $x = 3$ through the point $(2, 4)$ with a slope of the declared value.
  - The second point cannot be earned with a linear approximation using a slope other than $-\frac{4}{19}$ if that slope has not been declared to be the value of $\left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(2,\,4)}$.
- The second point cannot be earned for an approximation at any value of $x$ other than 3.
- A response does not have to write the tangent line equation but must clearly demonstrate its use at $x = 3$ in finding the requested approximation in order to earn both points.
- The minimal work required to earn both points is $4 - \frac{4}{19}(3-2)$.

**Total for part (a) 2 points**





**(b)** Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

| | | |
|---|---|---|
| $\frac{dy}{dx} = \frac{-2x}{3+4y} = 0 \Rightarrow x = 0$ | Considers $\frac{dy}{dx} = 0$ | **1 point** |
| And so, if the horizontal line $y = 1$ is tangent to the curve, the point of tangency must be $(0, 1)$. <br><br> However, the point $(0, 1)$ is not on the curve, because $0^2 + 3 \cdot 1 + 2 \cdot 1^2 = 5 \neq 48$. <br><br> Therefore, the horizontal line $y = 1$ is not tangent to the curve. | Answer with reason | **1 point** |

**Scoring notes:**

- The first point can be earned with a response of $\frac{dy}{dx} = 0$, $-2x = 0$, or $x = 0$, OR by identifying the point $(0, 1)$.
- To earn the second point a response must provide a reason that the line $y = 1$ is not tangent to the curve. Merely stating "$(0, 1)$ does not lie on the curve" is not sufficient to earn the second point.
- Alternate solution:

  If $y = 1$, then $x^2 + 3 \cdot 1 + 2 \cdot 1^2 = 48 \Rightarrow x = \pm\sqrt{43}$.

  $\left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(\pm\sqrt{43},\,1)} = \frac{\pm 2\sqrt{43}}{7} \neq 0$

  Therefore, the horizontal line $y = 1$ is not tangent to the curve.

  - A response that uses this method earns the first point by using $y = 1$ in $x^2 + 3y + 2y^2 = 48$.
  - A response that fails to consider both $x = +\sqrt{43}$ and $x = -\sqrt{43}$ does not earn the second point.

**Total for part (b) 2 points**

**(c)** The curve intersects the positive $x$-axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

| | | |
|---|---|---|
| At the point $(\sqrt{48}, 0)$, the slope of the line tangent to the curve is $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3+4(0)}$. The denominator of $\frac{dy}{dx}$ is $3+4(0)$, which does not equal $0$. Therefore, the line tangent to the curve at this point is not vertical. | Answer with reason | **1 point** |

**Scoring notes:**

- A response does not need to consider the numerator of $\left.\frac{dy}{dx}\right|_{(x,\, y)=(\sqrt{48},\, 0)}$ in order to earn this point; considering the denominator is sufficient.
- To earn this point a response must clearly demonstrate that the slope of the tangent line at the point $(\sqrt{48}, 0)$ is defined and answer "no."
  - Such demonstrations include, but are not limited to, the following:
    - $3+4(0) \neq 0$
    - At $(\sqrt{48}, 0)$, $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3}$.
    - $\frac{-2\sqrt{48}}{3}$
    - When $3+4y=0$, $y \neq 0$.

**Total for part (c)** **1 point**





**(d)** For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the $y$-coordinate of the particle's position is decreasing at a rate of $2$ units per second. At that instant, what is the rate of change of the $x$-coordinate of the particle's position with respect to time?

| | | |
|---|---|---|
| $3y^2\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$ | Attempts implicit differentiation | **1 point** |
| | $3y^2\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$ | **1 point** |
| $\frac{dy}{dt} = -2$ | Uses $\frac{dy}{dt} = \pm 2$ | **1 point** |
| $3(2)^2(-2) + 2(4)(-2) + 2(2)\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{40}{4} = 10$<br>The rate of change with respect to time in the $x$-coordinate is $10$ units per second. | Answer | **1 point** |

**Scoring notes:**

- The first point is earned for implicitly differentiating $y^3 + 2xy = 24$ with respect to $t$ with at most one error.
  - The first point can also be earned by correctly differentiating with respect to $x$: $3y^2\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$.
- The second point is earned for an equation equivalent to $3y^2\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$.
- A response does not need to explicitly declare $\frac{dy}{dt} = -2$ or $\frac{dy}{dt} = 2$ in order to earn the third point; this point may be earned by correctly substituting $\frac{dy}{dt} = -2$ or $\frac{dy}{dt} = 2$ in the implicitly differentiated equation. However, a response that uses both $\frac{dy}{dt} = -2$ and $\frac{dy}{dt} = 2$ in the implicitly differentiated equation does not earn the third point.
- The fourth point cannot be earned without the first 3 points. The fourth point is earned only for the value of $10$ with no mistakes in supporting work.
  - Note that a response that uses $\frac{dy}{dt} = 2$ and then mishandles subtracting $40$ from both sides of the equation, e.g., $3(2)^2(2) + 2(4)(2) + 2(2)\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{40}{4}$, does not earn the fourth point.

**Total for part (d)** **4 points**

**Total for question 5** **9 points**

## Part B (AB): Graphing calculator not allowed
## Question 6 — 9 points

**General Scoring Notes**
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve given by the equation $6xy = 2 + y^3$.

| Model Solution | Scoring | |
|---|---|---|
| **(a)** Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$. | | |
| $\frac{d}{dx}(6xy) = \frac{d}{dx}\left(2 + y^3\right) \Rightarrow 6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$ | Implicit differentiation | **1 point** |
| $\Rightarrow 2y = \frac{dy}{dx}\left(y^2 - 2x\right) \Rightarrow \frac{dy}{dx} = \frac{2y}{y^2 - 2x}$ | Verification | **1 point** |

**Scoring notes:**

- The first point is earned only for the correct implicit differentiation of $6xy = 2 + y^3$. Responses may use alternative notations for $\frac{dy}{dx}$, such as $y'$.
- The second point cannot be earned without the first point.
- It is sufficient to present $2y = \frac{dy}{dx}\left(y^2 - 2x\right)$ to earn the second point, provided there are no subsequent errors.

**Total for part (a) — 2 points**

**(b)** Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

| | | |
|---|---|---|
| For the line tangent to the curve to be horizontal, it is necessary that $2y = 0$ (so $y = 0$) and that $y^2 - 2x \neq 0$. | Sets $2y = 0$ | **1 point** |
| Substituting $y = 0$ into $6xy = 2 + y^3$ yields the equation $6x \cdot 0 = 2$, which has no solution.<br><br>Therefore, there is no point on the curve at which the line tangent to the curve is horizontal. | Answer with reason | **1 point** |





**Scoring notes:**

- The first point is earned with any of $2y = 0$, $y = 0$, $\frac{dy}{dx} = 0$, $dy = 0$, $y' = 0$, or $\frac{2y}{y^2 - 2x} = 0$.
- A response need not state that at a horizontal tangent, $y^2 - 2x \neq 0$.

**Total for part (b) — 2 points**

**(c)** Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

| | | |
|---|---|---|
| For a line tangent to this curve to be vertical, it is necessary that $2y \neq 0$ and that $y^2 - 2x = 0$ (so $x = \frac{y^2}{2}$). | Sets $y^2 - 2x = 0$ | **1 point** |
| Substituting $x = \frac{y^2}{2}$ into $6xy = 2 + y^3$ yields the equation $3y^2 \cdot y = 2 + y^3 \Rightarrow 2y^3 = 2 \Rightarrow y = 1$. | Substitutes $x = \frac{y^2}{2}$ into $6xy = 2 + y^3$ | **1 point** |
| Substituting $y = 1$ in $6xy = 2 + y^3$ yields $6x = 2 + 1$, or $x = \frac{1}{2}$.<br><br>The tangent line to the curve is vertical at the point $\left(\frac{1}{2}, 1\right)$. | Answer | **1 point** |

**Scoring notes:**

- The first point can be earned by presenting $y^2 = 2x$ or $y = \sqrt{2x}$.
- The second point can be earned for the substitution of $y = \sqrt{2x}$ into $6xy = 2 + y^3$, or for substituting $x = \frac{2 + y^3}{6y}$ into $y^2 - 2x = 0$.
- A response earns all three points by setting $y^2 - 2x = 0$, declaring the point $\left(\frac{1}{2}, 1\right)$, and verifying that this point is on the curve $6xy = 2 + y^3$.
- A response that identifies the point $\left(\frac{1}{2}, 1\right)$ but does not verify that the point is on the curve, does not earn the second or the third point.
- To earn the third point the response must present both coordinates of the point $\left(\frac{1}{2}, 1\right)$. The coordinates need not appear as an ordered pair as long as they are labeled.

**Total for part (c) — 3 points**

**(d)** A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ unit per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

| | | |
|---|---|---|
| $6y\frac{dx}{dt} + 6x\frac{dy}{dt} = 0 + 3y^2\frac{dy}{dt}$ | Uses implicit differentiation with respect to $t$ | **1 point** |
| At the point $(x, y) = \left(\frac{1}{2}, -2\right)$,<br>$6(-2)\left(\frac{2}{3}\right) + 6\left(\frac{1}{2}\right)\frac{dy}{dt} = 3(-2)^2\frac{dy}{dt}$<br>$\Rightarrow -8 + 3\frac{dy}{dt} = 12\frac{dy}{dt}$<br>$\Rightarrow \frac{dy}{dt} = -\frac{8}{9}$ unit per second | Answer | **1 point** |

**Scoring notes:**

- The first point is earned by presenting one or more of the terms $6y\frac{dx}{dt}$, $6x\frac{dy}{dt}$, or $3y^2\frac{dy}{dt}$.
- Units will not affect scoring in this part.
- An unsupported response of $-\frac{8}{9}$ earns no points.
- Alternate solution:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\left.\frac{dy}{dx}\right|_{(x,\,y)=(1/2,\,-2)} = \frac{2(-2)}{(-2)^2 - 2(1/2)} = -\frac{4}{3}$$

$$\left.\frac{dy}{dt}\right|_{(x,\,y)=(1/2,\,-2)} = \left.\frac{dy}{dx} \cdot \frac{dx}{dt}\right|_{(x,\,y)=(1/2,\,-2)} = -\frac{4}{3} \cdot \frac{2}{3} = -\frac{8}{9} \text{ unit per second}$$

  - The first point is earned for the statement $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or equivalent.
  - A numerical expression, such as $-\frac{4}{3} \cdot \frac{2}{3}$ or $\frac{2(-2)}{(-2)^2 - 2\left(\frac{1}{2}\right)} \cdot \frac{2}{3}$, earns both points.

| | |
|---|---|
| **Total for part (d)** | **2 points** |
| **Total for question 6** | **9 points** |





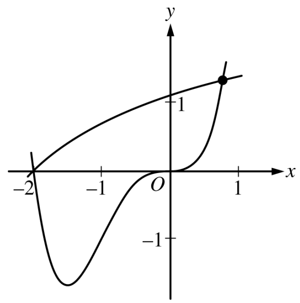
## Part A (AB): Graphing calculator required

## Question 2 — 9 points

**General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.




Let $f$ and $g$ be the functions defined by $f(x) = \ln(x + 3)$ and $g(x) = x^4 + 2x^3$. The graphs of $f$ and $g$, shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.

| Model Solution | Scoring | |
|---|---|---|
| **(a)** Find the area of the region enclosed by the graphs of $f$ and $g$. | | |
| $\ln(x+3) = x^4 + 2x^3 \Rightarrow x = -2,\ x = B = 0.781975$<br>$\int_{-2}^{B} (f(x) - g(x))\,dx = 3.603548$<br>The area of the region is $3.604$ (or $3.603$). | Integrand | **1 point** |
| | Limits of integration | **1 point** |
| | Answer | **1 point** |

**Scoring notes:**

- Other forms of the integrand in a definite integral, e.g., $|f(x) - g(x)|$, $|g(x) - f(x)|$, or $g(x) - f(x)$, earn the first point.
- To earn the second point, the response must have a lower limit of $-2$ and an upper limit expressed as either the letter $B$ with no value attached, or a number that is correct to the number of digits presented, with at least one and up to three decimal places.
    - Case 1: If the response did not earn the second point because of an incorrect value of $B$, $0 < B < 1$, but used a lower limit of $-2$, the response earns the third point only for a consistent answer.
    - Case 2: If the response did not earn the second point because the lower limit used was $x = 0$, but the response used a correct upper limit of $B$, the response earns the third point for a consistent answer of $0.708$ (or $0.707$).
    - Case 3: If a response uses any other incorrect limits it does not earn the second or third points.
- A response containing the integrand $g(x) - f(x)$ must interpret the value of the resulting integral correctly to earn the third point. For example, the following response earns all 3 points: $\int_{-2}^{B}(g(x) - f(x))\,dx = -3.604$ so the area is $3.604$. However, the response "Area $= \int_{-2}^{B}(g(x) - f(x))\,dx = 3.604$" presents an untrue statement and earns the first and second points but not the third point.
- A response must earn the first point in order to be eligible for the third point. If the response has earned the second point, then only the correct answer will earn the third point.
- Instructions for scoring a response that presents an integrand of $\ln(x+3) - x^4 + 2x^3$ and the correct answer are shown in the "Global Special Case" after part (d).

**Total for part (a) 3 points**





**(b)** For $-2 \le x \le B$, let $h(x)$ be the vertical distance between the graphs of $f$ and $g$. Is $h$ increasing or decreasing at $x = -0.5$? Give a reason for your answer.

| | | |
|---|---|---|
| $h(x) = f(x) - g(x)$<br>$h'(x) = f'(x) - g'(x)$<br>$h'(-0.5) = f'(-0.5) - g'(-0.5) = -0.6$ (or $-0.599$) | Considers $h'(-0.5)$<br>– OR –<br>$f'(x) - g'(x)$ | **1 point** |
| Since $h'(-0.5) < 0$, $h$ is decreasing at $x = -0.5$. | Answer with reason | **1 point** |

**Scoring notes:**

- The response need not present the value of $h'(-0.5)$. The last line earns both points. However, if a value is presented it must be correct for the digits reported up to three decimal places.
- A response that reports an incorrect value of $h'(-0.5)$ earns only the first point.
- A response that presents only $h'(x)$ does not earn either point.
- The only response that earns the second point for concluding "$h$ is increasing" is described in the "Global Special Case" provided after part (d).
- A response that compares the values of $f'(x)$ and $g'(x)$ at $x = -0.5$ earns the first point and is eligible for the second point. This comparison can be made symbolically or verbally; for example, the response "the rate of change of $f(x)$ is less than the rate of change of $g(x)$ at $x = -0.5$" earns the first point.

**Total for part (b) 2 points**

**(c)** The region enclosed by the graphs of $f$ and $g$ is the base of a solid. Cross sections of the solid taken perpendicular to the $x$-axis are squares. Find the volume of the solid.

| | | |
|---|---|---|
| $\int_{-2}^{B}(f(x)-g(x))^2\,dx = 5.340102$ | Integrand | **1 point** |
| The volume of the solid is $5.340$. | Answer | **1 point** |

**Scoring notes:**

- The first point is earned for an integrand of $k(f(x)-g(x))^2$ or its equivalent with $k \neq 0$ in any definite integral. If $k \neq 1$, then the response is not eligible for the second point.
- A response that does not earn the first point is ineligible to earn the second point, with the following exceptions:
  - A response which has a presentation error in the integrand (for example, mismatched or missing parentheses, misplaced exponent) does not earn the first point but would earn the second point for the correct answer. A response which has a presentation error in the integrand and which reports an incorrect answer earns no points.
  - A response that presents an integrand of $\left(\ln(x+3)-x^4+2x^3\right)^2$. Scoring instructions for this case are provided in the "Global Special Case" after part (d).
- A response that uses incorrect limits is only eligible for the second point, provided the limits are imported from part (a) in Case 1 or Case 2. In both of these situations, the second point is earned only for answers consistent with the imported limits.

**Total for part (c) 2 points**

**(d)** A vertical line in the $xy$-plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of $7$ units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$.

| | | |
|---|---|---|
| The cross section has area $A(x) = (f(x)-g(x))^2$.<br><br>$\frac{d}{dt}[A(x)] = \frac{dA}{dx}\cdot\frac{dx}{dt}$ | $\frac{dA}{dx}\cdot\frac{dx}{dt}$ | **1 point** |
| $\frac{d}{dt}[A(x)]\Big\vert_{x=-0.5} = A'(-0.5)\cdot 7 = -9.271842$<br><br>At $x=-0.5$, the area of the cross section above the line is changing at a rate of $-9.272$ (or $-9.271$) square units per second. | Answer | **1 point** |





**Scoring notes:**

- The first point may be earned by presenting $\frac{dA}{dx}\cdot\frac{dx}{dt}$, $A'\cdot\frac{dx}{dt}$, $A'(x)\cdot\frac{dx}{dt}$, $A'(x)\cdot x'$, $A'(-0.5)\cdot 7$, or $k\cdot 7$, where $k$ is a declared value of $A'(-0.5)$, or any equivalent expression, including $2(f(x)-g(x))(f'(x)-g'(x))\frac{dx}{dt}$.
- If a response defines $f(x)-g(x)$ as a function in parts (b) or (c) (for example, $h(x) = f(x)-g(x)$), then a correct expression for $\frac{dA}{dt}$ (for example, $2h\frac{dh}{dt}$) earns the first point.
- A response that imports a function $A(x)$ declared in part (c) is eligible for both points (the answer must be consistent with the imported function $A(x)$).
- A response that presents an incorrect function for $A(x)$ that is not imported from part (c) is eligible only for the first point.
- Except when $A(x)$ is imported from part (c), the second point is earned only for the correct answer.
- A response that does not earn the first point is ineligible to earn the second point except in the special case noted below.

**Total for part (d) 2 points**

**Total for question 2 9 points**

Global Special Case: A response may incorrectly simplify $f(x)-g(x)$ to $j(x) = \ln(x+3)-x^4+2x^3$ instead of $\ln(x+3)-x^4-2x^3$. Because this question is calculator active, a response with this incorrect simplification may nevertheless present correct answers.

- In any part of the question, a response that starts correctly by using $f(x)-g(x)$, then presents $j(x)$, is eligible for all points in that part.
- The first time a response implicitly presents $f(x)-g(x)$ as $j(x) = \ln(x+3)-x^4+2x^3$ (with no explicit connection) in any part of this question, the response loses a point. The response is then eligible for all remaining points for a correct or consistent answer.
- In part (a) the consistent answer using $j(x)$ is negative and will not earn the third point.
- In part (b) the consistent answer using $j(x)$ is that $j'(-0.5) = 2.4 > 0$, so $h$ is increasing at $x = -0.5$.
- In part (c) the consistent answer using $j(x)$ is $252.187$ (or $252.188$).
- In part (d) the consistent answer using $j(x)$ is $20.287$.

## Part B (AB or BC): Graphing calculator not allowed
## Question 4 — 9 points

**General Scoring Notes**
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

| $t$ (days) | 0 | 3 | 7 | 10 | 12 |
|---|---|---|---|---|---|
| $r'(t)$ (centimeters per day) | $-6.1$ | $-5.0$ | $-4.4$ | $-3.8$ | $-3.5$ |

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function $r$, where $r(t)$ is measured in centimeters and $t$ is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

| Model Solution | Scoring | |
|---|---|---|

**(a)** Approximate $r''(8.5)$ using the average rate of change of $r'$ over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

| | | |
|---|---|---|
| $r''(8.5) \approx \dfrac{r'(10) - r'(7)}{10 - 7} = \dfrac{-3.8 - (-4.4)}{10 - 7}$ | $r''(8.5)$ with supporting work | **1 point** |
| $= \dfrac{0.6}{3} = 0.2$ centimeter per day per day | Units | **1 point** |

**Scoring notes:**

- To earn the first point the supporting work must include at least a difference and a quotient.
- Simplification is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The second point can be earned with an incorrect approximation for $r''(8.5)$ but cannot be earned without some value for $r''(8.5)$ presented.
- Units may be written in any equivalent form (such as $\text{cm}/\text{day}^2$ ).

**Total for part (a) — 2 points**





**(b)** Is there a time $t$, $0 \le t \le 3$, for which $r'(t) = -6$ ? Justify your answer.

| | | |
|---|---|---|
| $r(t)$ is twice-differentiable. $\Rightarrow$ $r'(t)$ is differentiable. $\Rightarrow$ $r'(t)$ is continuous. | $r'(0) < -6 < r'(3)$ | **1 point** |
| $r'(0) = -6.1 < -6 < -5.0 = r'(3)$ Therefore, by the Intermediate Value Theorem, there is a time $t$, $0 < t < 3$, such that $r'(t) = -6$. | Conclusion using Intermediate Value Theorem | **1 point** |

**Scoring notes:**

- To earn the first point, the response must establish that $-6$ is between $r'(0)$ and $r'(3)$ (or $-6.1$ and $-5$ ). This statement may be represented symbolically (with or without including one or both endpoints in an inequality) or verbally. A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not state that $-6$ is between $-6.1$ and $-5$. Thus this response does not earn the first point.
- To earn the second point:
  - The response must state that $r'(t)$ is continuous because $r'(t)$ is differentiable (or because $r(t)$ is twice differentiable).
  - The response must have earned the first point.
    - Exception: A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not earn the first point because of imprecise communication but may nonetheless earn the second point if all other criteria for the second point are met.
  - The response must conclude that there is a time $t$ such that $r'(t) = -6$. (A statement of "yes" would be sufficient.)
- To earn the second point, the response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

**Total for part (b) — 2 points**

**(c)** Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

| | | |
|---|---|---|
| $\int_0^{12} r'(t)\,dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$ | Form of right Riemann sum | **1 point** |
| $= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$<br>$= -51$ | Answer | **1 point** |

**Scoring notes:**

- To earn the first point, at least seven of the eight factors in the Riemann sum must be correct. If there is any error in the Riemann sum, the response does not earn the second point.
- A response of $3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A response that presents the correct answer, with accompanying work that shows the four products in the Riemann sum (without explicitly showing all of the factors and/or the sum process) does not earn the first point but earns the second point. For example, $-15 + 4(-4.4) + 3(-3.8) + -7$ does not earn the first point but earns the second point. Similarly, $-15, -17.6, -11.4, -7 \rightarrow -51$ does not earn the first point but earns the second point.
- A response that presents the correct answer ($-51$) with no supporting work earns no points.
- A response that provides a completely correct left Riemann sum and approximation $\int_0^{12} r'(t)\,dt$ (i.e., $3r'(0) + 4r'(3) + 3r'(7) + 2r'(10) = 3(-6.1) + 4(-5.0) + 3(-4.4) + 2(-3.8) = -59.1$) earns 1 of the 2 points. A response that has any error in a left Riemann sum or evaluation for $\int_0^{12} r'(t)\,dt$ earns no points.
- Units are not required or read in this part.

**Total for part (c) 2 points**





**(d)** The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume $V$ of a cone with radius $r$ and height $h$ is $V = \frac{1}{3}\pi r^2 h$.)

| | | |
|---|---|---|
| $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}$ | Product rule | **1 point** |
| | Chain rule | **1 point** |
| $\left.\frac{dV}{dt}\right\vert_{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\frac{70{,}000\pi}{3}$<br><br>The rate of change of the volume of the sculpture at $t = 3$ is approximately $-\frac{70{,}000\pi}{3}$ cubic centimeters per day. | Answer | **1 point** |

**Scoring notes:**

- The first 2 points could be earned in either order.
- A response with a completely correct product rule, missing one or both of the correct differentials, earns the product rule point, but not the chain rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh + \frac{1}{3}\pi r^2$ earns the first point, but not the second.
- A response that treats $r$ or $h$ (but not both) as a constant is eligible for the chain rule point but not the product rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}$ is correct if $h$ is constant, and thus earns the chain rule point.
- Note: Neither $\frac{dV}{dt} = \frac{2}{3}\pi r\frac{dh}{dt}$ nor $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}\frac{dh}{dt}$ earns any points.
- A response that assumes a functional relationship between $r$ and $h$ (such as $r = 2h$), and uses this relationship to create a function for volume in terms of one variable, is eligible for at most the chain rule point. For example, $r = 2h \rightarrow V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3 \rightarrow \frac{dV}{dt} = 4\pi h^2\frac{dh}{dt}$ earns only the chain rule point.
- A response that mishandles the constant $\frac{1}{3}\pi$ cannot earn the third point but is eligible for the first 2 points.
- The third point cannot be earned without both of the first 2 points.
- $\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2)$ earns all 3 points.
- Units are not required or read in this part.

**Total for part (d) 3 points**

**Total for question 4 9 points**

2019

AP® CollegeBoard

# AP® Calculus AB
## Scoring Guidelines





## 2019 SCORING GUIDELINES

### Question 1

| | |
|---|---|
| (a) $\int_0^5 E(t)\,dt = 153.457690$<br><br>To the nearest whole number, $153$ fish enter the lake from midnight to $5$ A.M. | $2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$ |
| (b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$<br><br>The average number of fish that leave the lake per hour from midnight to $5$ A.M. is $6.059$ fish per hour. | $2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$ |
| (c) The rate of change in the number of fish in the lake at time $t$ is given by $E(t) - L(t)$.<br><br>$E(t) - L(t) = 0 \Rightarrow t = 6.20356$<br><br>$E(t) - L(t) > 0$ for $0 \le t < 6.20356,$ and $E(t) - L(t) < 0$ for $6.20356 < t \le 8.$ Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or $6.203$).<br><br>— OR —<br><br>Let $A(t)$ be the change in the number of fish in the lake from midnight to $t$ hours after midnight.<br><br>$A(t) = \int_0^t (E(s) - L(s))\,ds$<br>$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$<br><br>$t$ \| $A(t)$: $0$ \| $0$; $C$ \| $135.01492$; $8$ \| $80.91998$<br><br>Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or $6.203$). | $3: \begin{cases} 1: \text{sets } E(t) - L(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$ |
| (d) $E'(5) - L'(5) = -10.7228 < 0$<br><br>Because $E'(5) - L'(5) < 0,$ the rate of change in the number of fish is decreasing at time $t = 5.$ | $2: \begin{cases} 1: \text{considers } E'(5) \text{ and } L'(5) \\ 1: \text{answer with explanation} \end{cases}$ |

Table in part (c):

| $t$ | $A(t)$ |
|---|---|
| $0$ | $0$ |
| $C$ | $135.01492$ |
| $8$ | $80.91998$ |

## Question 2

(a) $v_P$ is differentiable $\Rightarrow$ $v_P$ is continuous on $0.3 \le t \le 2.8$.

$$\frac{v_P(2.8) - v_P(0.3)}{2.8 - 0.3} = \frac{55 - 55}{2.5} = 0$$

By the Mean Value Theorem, there is a value $c$, $0.3 < c < 2.8$, such that $v_P{}'(c) = 0$.

— OR —

$v_P$ is differentiable $\Rightarrow$ $v_P$ is continuous on $0.3 \le t \le 2.8$.

By the Extreme Value Theorem, $v_p$ has a minimum on $[0.3, 2.8]$.

$v_P(0.3) = 55 > -29 = v_P(1.7)$ and $v_P(1.7) = -29 < 55 = v_P(2.8)$.

Thus $v_P$ has a minimum on the interval $(0.3, 2.8)$.

Because $v_P$ is differentiable, $v_P{}'(t)$ must equal $0$ at this minimum.

$$2: \begin{cases} 1: v_P(2.8) - v_P(0.3) = 0 \\ 1: \text{justification, using Mean Value Theorem} \end{cases}$$

— OR —

$$2: \begin{cases} 1: v_P(0.3) > v_P(1.7) \text{ and } v_P(1.7) < v_P(2.8) \\ 1: \text{justification, using Extreme Value Theorem} \end{cases}$$

(b) 
$$\int_0^{2.8} v_P(t)\,dt \approx 0.3\left(\frac{v_P(0) + v_P(0.3)}{2}\right) + 1.4\left(\frac{v_P(0.3) + v_P(1.7)}{2}\right) + 1.1\left(\frac{v_P(1.7) + v_P(2.8)}{2}\right)$$
$$= 0.3\left(\frac{0 + 55}{2}\right) + 1.4\left(\frac{55 + (-29)}{2}\right) + 1.1\left(\frac{-29 + 55}{2}\right)$$
$$= 40.75$$

1 : answer, using trapezoidal sum

(c) $v_Q(t) = 60 \Rightarrow t = A = 1.866181$ or $t = B = 3.519174$

$v_Q(t) \ge 60$ for $A \le t \le B$

$$\int_A^B \left|v_Q(t)\right|\,dt = 106.108754$$

The distance traveled by particle $Q$ during the interval $A \le t \le B$ is $106.109$ (or $106.108$) meters.

$$3: \begin{cases} 1: \text{interval} \\ 1: \text{definite integral} \\ 1: \text{distance} \end{cases}$$

(d) From part (b), the position of particle $P$ at time $t = 2.8$ is

$$x_P(2.8) = \int_0^{2.8} v_P(t)\,dt \approx 40.75.$$

$$x_Q(2.8) = x_Q(0) + \int_0^{2.8} v_Q(t)\,dt = -90 + 135.937653 = 45.937653$$

Therefore at time $t = 2.8$, particles $P$ and $Q$ are approximately $45.937653 - 40.75 = 5.188$ (or $5.187$) meters apart.

$$3: \begin{cases} 1: \int_0^{2.8} v_Q(t)\,dt \\ 1: \text{position of particle } Q \\ 1: \text{answer} \end{cases}$$





## Question 3

(a) 
$$\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$$
$$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$$
$$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$$

$$3: \begin{cases} 1: \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1: \int_{-2}^{5} f(x)\,dx \\ 1: \text{answer} \end{cases}$$

(b) 
$$\int_3^5 (2f'(x) + 4)\,dx = 2\int_3^5 f'(x)\,dx + \int_3^5 4\,dx$$
$$= 2(f(5) - f(3)) + 4(5 - 3)$$
$$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$$
$$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$$

— OR —

$$\int_3^5 (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$$
$$= (2f(5) + 20) - (2f(3) + 12)$$
$$= (2 \cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$$
$$= 2 + 2\sqrt{5}$$

$$2: \begin{cases} 1: \text{Fundamental Theorem of Calculus} \\ 1: \text{answer} \end{cases}$$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

| $x$ | $g(x)$ |
|---|---|
| $-2$ | $0$ |
| $-1$ | $\frac{1}{2}$ |
| $\frac{1}{2}$ | $-\frac{1}{4}$ |
| $5$ | $11 - \frac{9\pi}{4}$ |

On the interval $-2 \le x \le 5$, the absolute maximum value of $g$ is $g(5) = 11 - \frac{9\pi}{4}$.

$$3: \begin{cases} 1: g'(x) = f(x) \\ 1: \text{identifies } x = -1 \text{ as a candidate} \\ 1: \text{answer with justification} \end{cases}$$

(d) 
$$\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$$
$$= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$$

1 : answer

## 2019 SCORING GUIDELINES

### Question 4

(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$

$\left.\dfrac{dV}{dt}\right|_{h=4} = \pi \left.\dfrac{dh}{dt}\right|_{h=4} = \pi\left(-\dfrac{1}{10}\sqrt{4}\right) = -\dfrac{\pi}{5}$ cubic feet per second

$2 : \begin{cases} 1 : \dfrac{dV}{dt} = \pi \dfrac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$

(b) $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} = -\dfrac{1}{20\sqrt{h}} \cdot \left(-\dfrac{1}{10}\sqrt{h}\right) = \dfrac{1}{200}$

Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0,$ the rate of change of the height is increasing when the height of the water is 3 feet.

$3 : \begin{cases} 1 : \dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}} \\ 1 : \dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$

(c) $\dfrac{dh}{\sqrt{h}} = -\dfrac{1}{10}\,dt$

$\displaystyle\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\dfrac{1}{10}t + C$

$2\sqrt{5} = -\dfrac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$

$2\sqrt{h} = -\dfrac{1}{10}t + 2\sqrt{5}$

$h(t) = \left(-\dfrac{1}{20}t + \sqrt{5}\right)^2$

$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration





## 2019 SCORING GUIDELINES

### Question 5

(a) $\displaystyle\int_0^2 (h(x) - g(x))\,dx = \int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right) dx$

$\displaystyle = \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$

$\displaystyle = \left(\left(12 - \frac{2}{3}\right) - (-4 + 0)\right) - \left(\left(0 + \frac{2}{3}\right) - (0 + 0)\right)$

$\displaystyle = 12 - \frac{2}{3} + 4 - \frac{2}{3} = \frac{44}{3}$

The area of $R$ is $\dfrac{44}{3}.$

$4 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative of } 3\cos\left(\frac{\pi}{2}x\right) \\ 1 : \text{antiderivative of} \\ \quad\ \text{remaining terms} \\ 1 : \text{answer} \end{cases}$

(b) $\displaystyle\int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx$

$= \left[\ln(x+3)\right]_{x=0}^{x=2} = \ln 5 - \ln 3$

The volume of the solid is $\ln 5 - \ln 3.$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $\displaystyle\pi\int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2\right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{form of integrand} \\ 1 : \text{integrand} \end{cases}$

## 2019 SCORING GUIDELINES

### Question 6

(a) $h'(2) = \frac{2}{3}$

1 : answer

(b) $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$

$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$

$3 : \begin{cases} 1 : \text{form of product rule} \\ 1 : a'(x) \\ 1 : a'(2) \end{cases}$

(c) Because $h$ is differentiable, $h$ is continuous, so $\lim\limits_{x \to 2} h(x) = h(2) = 4$.

Also, $\lim\limits_{x \to 2} h(x) = \lim\limits_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3}$, so $\lim\limits_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$.

Because $\lim\limits_{x \to 2} \left(x^2 - 4\right) = 0$, we must also have $\lim\limits_{x \to 2} \left(1 - (f(x))^3\right) = 0$.

Thus $\lim\limits_{x \to 2} f(x) = 1$.

Because $f$ is differentiable, $f$ is continuous, so $f(2) = \lim\limits_{x \to 2} f(x) = 1$.

Also, because $f$ is twice differentiable, $f'$ is continuous, so $\lim\limits_{x \to 2} f'(x) = f'(2)$ exists.

Using L'Hospital's Rule,

$$\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$$

Thus $f'(2) = -\frac{1}{3}$.

$4 : \begin{cases} 1 : \lim\limits_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4 \\ 1 : f(2) \\ 1 : \text{L'Hospital's Rule} \\ 1 : f'(2) \end{cases}$

(d) Because $g$ and $h$ are differentiable, $g$ and $h$ are continuous, so $\lim\limits_{x \to 2} g(x) = g(2) = 4$ and $\lim\limits_{x \to 2} h(x) = h(2) = 4$.

Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim\limits_{x \to 2} k(x) = 4$.

Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$.

Thus $k$ is continuous at $x = 2$.

1 : continuous with justification





2018

# AP Calculus AB

## Scoring Guidelines

# 2018 SCORING GUIDELINES

## Question 1

(a) $\int_0^{300} r(t)\,dt = 270$

According to the model, 270 people enter the line for the escalator during the time interval $0 \le t \le 300$.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $20 + \int_0^{300} (r(t) - 0.7)\,dt = 20 + \int_0^{300} r(t)\,dt - 0.7 \cdot 300 = 80$

According to the model, 80 people are in line at time $t = 300$.

$2 : \begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$

(c) Based on part (b), the number of people in line at time $t = 300$ is 80.

The first time $t$ that there are no people in line is

$300 + \frac{80}{0.7} = 414.286$ (or 414.285) seconds.

1 : answer

(d) The total number of people in line at time $t$, $0 \le t \le 300$, is modeled by $20 + \int_0^t r(x)\,dx - 0.7t$.

$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298,\ t_2 = 166.574719$

| $t$ | People in line for escalator |
|---|---|
| 0 | 20 |
| $t_1$ | 3.803 |
| $t_2$ | 158.070 |
| 300 | 80 |

The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line.

$4 : \begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$





# 2018 SCORING GUIDELINES

## Question 2

(a) $v'(3) = -2.118$

The acceleration of the particle at time $t = 3$ is $-2.118$.

1 : answer

(b) $x(3) = x(0) + \int_0^3 v(t)\,dt = -5 + \int_0^3 v(t)\,dt = -1.760213$

The position of the particle at time $t = 3$ is $-1.760$.

$3 : \begin{cases} 1 : \int_0^3 v(t)\,dt \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^{3.5} v(t)\,dt = 2.844$ (or 2.843)

$\int_0^{3.5} |v(t)|\,dt = 3.737$

The integral $\int_0^{3.5} v(t)\,dt$ is the displacement of the particle over the time interval $0 \le t \le 3.5$.

The integral $\int_0^{3.5} |v(t)|\,dt$ is the total distance traveled by the particle over the time interval $0 \le t \le 3.5$.

$3 : \begin{cases} 1 : \text{answers} \\ 2 : \text{interpretations of } \int_0^{3.5} v(t)\,dt \\ \quad \text{and } \int_0^{3.5} |v(t)|\,dt \end{cases}$

(d) $v(t) = x_2'(t)$

$v(t) = 2t - 1 \Rightarrow t = 1.57054$

The two particles are moving with the same velocity at time $t = 1.571$ (or 1.570).

$2 : \begin{cases} 1 : \text{sets } v(t) = x_2'(t) \\ 1 : \text{answer} \end{cases}$

# AP® CALCULUS AB/CALCULUS BC
## 2018 SCORING GUIDELINES

### Question 3

(a) $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx$

$= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx$

$= \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx$

$= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$

$3 : \begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$

(c) The graph of $f$ is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(d) The graph of $f$ has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$






# AP® CALCULUS AB/CALCULUS BC
## 2018 SCORING GUIDELINES


### Question 4

(a) $H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$

Because $H$ is differentiable on $3 \le t \le 5$, $H$ is continuous on $3 \le t \le 5$.

By the Mean Value Theorem, there exists a value $c$, $3 < c < 5$, such that $H'(c) = 2$.

$2 : \begin{cases} 1 : \frac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$

(c) The average height of the tree over the time interval $2 \le t \le 10$ is given by $\frac{1}{10 - 2}\int_2^{10} H(t)\,dt$.

$$\frac{1}{8}\int_2^{10} H(t)\,dt \approx \frac{1}{8}\left(\frac{1.5 + 2}{2}\cdot 1 + \frac{2 + 6}{2}\cdot 2 + \frac{6 + 11}{2}\cdot 2 + \frac{11 + 15}{2}\cdot 3\right)$$
$$= \frac{1}{8}(65.75) = \frac{263}{32}$$

The average height of the tree over the time interval $2 \le t \le 10$ is $\frac{263}{32}$ meters.

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$

(d) $G(x) = 50 \Rightarrow x = 1$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x))\cdot\frac{dx}{dt} = \frac{(1 + x)100 - 100x\cdot 1}{(1 + x)^2}\cdot\frac{dx}{dt} = \frac{100}{(1 + x)^2}\cdot\frac{dx}{dt}$$

$$\left.\frac{d}{dt}(G(x))\right|_{x=1} = \frac{100}{(1 + 1)^2}\cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\frac{3}{4}$ meter per year.

$3 : \begin{cases} 2 : \frac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$

Note: max $1/3$ [1-0] if no chain rule

## 2018 SCORING GUIDELINES

### Question 5

(a) The average rate of change of $f$ on the interval $0 \le x \le \pi$ is $\dfrac{f(\pi) - f(0)}{\pi - 0} = \dfrac{-e^{\pi} - 1}{\pi}$.

1 : answer

(b) $f'(x) = e^x \cos x - e^x \sin x$

$f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$

The slope of the line tangent to the graph of $f$ at $x = \frac{3\pi}{2}$ is $e^{3\pi/2}$.

$2 : \begin{cases} 1 : f'(x) \\ 1 : \text{slope} \end{cases}$

(c) $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \frac{\pi}{4},\ x = \frac{5\pi}{4}$

| $x$ | $f(x)$ |
|---|---|
| $0$ | $1$ |
| $\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}} e^{\pi/4}$ |
| $\frac{5\pi}{4}$ | $-\frac{1}{\sqrt{2}} e^{5\pi/4}$ |
| $2\pi$ | $e^{2\pi}$ |

The absolute minimum value of $f$ on $0 \le x \le 2\pi$ is $-\frac{1}{\sqrt{2}} e^{5\pi/4}$.

$3 : \begin{cases} 1 : \text{sets } f'(x) = 0 \\ 1 : \text{identifies } x = \frac{\pi}{4},\ x = \frac{5\pi}{4} \\ \quad \text{as candidates} \\ 1 : \text{answer with justification} \end{cases}$

(d) $\lim\limits_{x \to \pi/2} f(x) = 0$

Because $g$ is differentiable, $g$ is continuous.

$\lim\limits_{x \to \pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$

By L'Hospital's Rule,

$\lim\limits_{x \to \pi/2} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \pi/2} \dfrac{f'(x)}{g'(x)} = \dfrac{-e^{\pi/2}}{2}$.

$3 : \begin{cases} 1 : g \text{ is continuous at } x = \frac{\pi}{2} \\ \quad \text{and limits equal 0} \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

Note: max 1/3 [1-0-0] if no limit notation attached to a ratio of derivatives

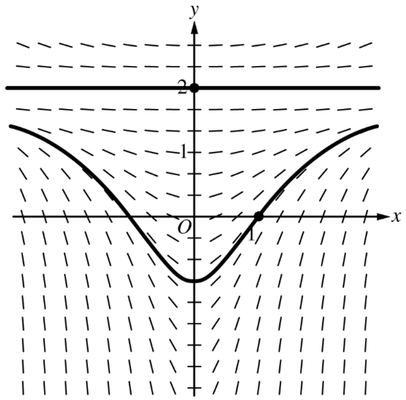




## 2018 SCORING GUIDELINES

### Question 6

(a)




$2 : \begin{cases} 1 : \text{solution curve through } (0, 2) \\ 1 : \text{solution curve through } (1, 0) \end{cases}$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

(b) $\left.\dfrac{dy}{dx}\right|_{(x,\,y)=(1,\,0)} = \dfrac{4}{3}$

An equation for the line tangent to the graph of $y = f(x)$ at $x = 1$ is $y = \frac{4}{3}(x - 1)$.

$f(0.7) \approx \frac{4}{3}(0.7 - 1) = -0.4$

$2 : \begin{cases} 1 : \text{equation of tangent line} \\ 1 : \text{approximation} \end{cases}$

(c) $\dfrac{dy}{dx} = \dfrac{1}{3}x(y-2)^2$

$\displaystyle\int \frac{dy}{(y-2)^2} = \int \frac{1}{3}x\,dx$

$\dfrac{-1}{y-2} = \dfrac{1}{6}x^2 + C$

$\dfrac{1}{2} = \dfrac{1}{6} + C \Rightarrow C = \dfrac{1}{3}$

$\dfrac{-1}{y-2} = \dfrac{1}{6}x^2 + \dfrac{1}{3} = \dfrac{x^2 + 2}{6}$

$y = 2 - \dfrac{6}{x^2 + 2}$

Note: this solution is valid for $-\infty < x < \infty$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad \text{and uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: 0/5 if no separation of variables

Note: max 3/5 [1-2-0-0] if no constant of integration

# AP® Calculus AB
# 2016 Scoring Guidelines







## 2016 SCORING GUIDELINES

### Question 1

| $t$ (hours) | 0 | 1 | 3 | 6 | 8 |
|---|---|---|---|---|---|
| $R(t)$ (liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where $t$ is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where $R$ is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time $t$ when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

---

(a) $R'(2) \approx \dfrac{R(3) - R(1)}{3 - 1} = \dfrac{950 - 1190}{3 - 1} = -120$ liters/hr$^2$

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by $\int_0^8 R(t)\,dt$.

$$\begin{aligned}\int_0^8 R(t)\,dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters}\end{aligned}$$

This is an overestimate since $R$ is a decreasing function.

$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $$\begin{aligned}\text{Total} &\approx 50000 + \int_0^8 W(t)\,dt - 8050 \\ &= 50000 + 7836.195325 - 8050 \approx 49786 \text{ liters}\end{aligned}$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

Therefore, the Intermediate Value Theorem guarantees at least one time $t$, $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of $t$, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

## 2016 SCORING GUIDELINES

### Question 2

For $t \geq 0$, a particle moves along the $x$-axis. The velocity of the particle at time $t$ is given by $v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right)$. The particle is at position $x = 2$ at time $t = 4$.

(a) At time $t = 4$, is the particle speeding up or slowing down?

(b) Find all times $t$ in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.

(c) Find the position of the particle at time $t = 0$.

(d) Find the total distance the particle travels from time $t = 0$ to time $t = 3$.

---

(a) $v(4) = 2.978716 > 0$
$v'(4) = -1.164000 < 0$

The particle is slowing down since the velocity and acceleration have different signs.

2 : conclusion with reason

(b) $v(t) = 0 \Rightarrow t = 2.707468$

$v(t)$ changes from positive to negative at $t = 2.707$. Therefore, the particle changes direction at this time.

$2 : \begin{cases} 1 : t = 2.707 \\ 1 : \text{justification} \end{cases}$

(c) $x(0) = x(4) + \int_4^0 v(t)\,dt$
$= 2 + (-5.815027) = -3.815$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(d) Distance $= \int_0^3 |v(t)|\,dt = 5.301$

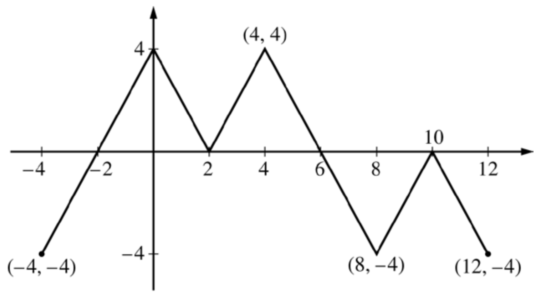
$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$





## 2016 SCORING GUIDELINES

### Question 3




Graph of $f$

The figure above shows the graph of the piecewise-linear function $f$. For $-4 \leq x \leq 12$, the function $g$ is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of $g$ have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of $g$ on the interval $-4 \leq x \leq 12$. Justify your answers.

(d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

---

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

(a) The function $g$ has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \leq 0$ for $8 \leq x \leq 12$.

1 : answer with justification

(b) The graph of $g$ has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \leq x \leq 4$ and decreasing for $4 \leq x \leq 8$.

1 : answer with justification

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

| $x$ | $g(x)$ |
|---|---|
| −4 | −4 |
| −2 | −8 |
| 6 | 8 |
| 12 | −4 |

On the interval $-4 \leq x \leq 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

$4 : \begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \\ \quad \text{as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$

(d) $g(x) \leq 0$ for $-4 \leq x \leq 2$ and $10 \leq x \leq 12$.

2 : intervals

# 2016 SCORING GUIDELINES

## Question 4

Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

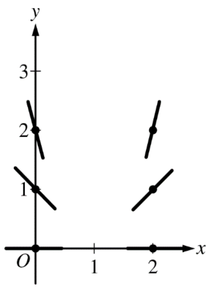
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

---

(a)




$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} = \frac{3^2}{2-1} = 9$

$2 : \begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

An equation for the tangent line is $y = 9(x-2) + 3$.

$f(2.1) \approx 9(2.1-2) + 3 = 3.9$

(c) $\frac{1}{y^2}\,dy = \frac{1}{x-1}\,dx$

$\int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx$

$-\frac{1}{y} = \ln|x-1| + C$

$-\frac{1}{3} = \ln|2-1| + C \Rightarrow C = -\frac{1}{3}$

$-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$

$y = \frac{1}{\frac{1}{3} - \ln(x-1)}$

Note: This solution is valid for $1 < x < 1 + e^{1/3}$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/5$ [1-2-0-0] if no constant of integration

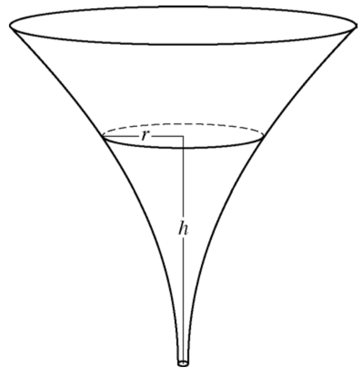
Note: $0/5$ if no separation of variables





# 2016 SCORING GUIDELINES

## Question 5




The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height $h$, the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$, where $0 \le h \le 10$. The units of $r$ and $h$ are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

---

(a) Average radius $= \frac{1}{10}\int_0^{10} \frac{1}{20}(3 + h^2)\,dh = \frac{1}{200}\left[3h + \frac{h^3}{3}\right]_0^{10}$

$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - 0\right) = \frac{109}{60}$ in

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi\int_0^{10} \left(\left(\frac{1}{20}\right)(3 + h^2)\right)^2 dh = \frac{\pi}{400}\int_0^{10} \left(9 + 6h^2 + h^4\right) dh$

$= \frac{\pi}{400}\left[9h + 2h^3 + \frac{h^5}{5}\right]_0^{10}$

$= \frac{\pi}{400}\left(\left(90 + 2000 + \frac{100000}{5}\right) - 0\right) = \frac{2209\pi}{40}$ $\text{in}^3$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$

$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{1}{5}\cdot\frac{10}{3} = -\frac{2}{3}$ in/sec

$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

## 2016 SCORING GUIDELINES

### Question 6

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | –6 | 3 | 2 | 8 |
| 2 | 2 | –2 | –3 | 0 |
| 3 | 8 | 7 | 6 | 2 |
| 6 | 4 | 5 | 3 | –1 |

The functions $f$ and $g$ have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of $x$.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of $k$ at $x = 3$.

(b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

---

(a) $k(3) = f(g(3)) = f(6) = 4$

$k'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot 2 = 5 \cdot 2 = 10$

An equation for the tangent line is $y = 10(x - 3) + 4$.

$3 : \begin{cases} 2 : \text{slope at } x = 3 \\ 1 : \text{equation for tangent line} \end{cases}$

(b) $$h'(1) = \frac{f(1) \cdot g'(1) - g(1) \cdot f'(1)}{(f(1))^2} = \frac{(-6) \cdot 8 - 2 \cdot 3}{(-6)^2} = \frac{-54}{36} = -\frac{3}{2}$$

$3 : \begin{cases} 2 : \text{expression for } h'(1) \\ 1 : \text{answer} \end{cases}$

(c) $$\int_1^3 f''(2x)\,dx = \frac{1}{2}\left[f'(2x)\right]_1^3 = \frac{1}{2}[f'(6) - f'(2)] = \frac{1}{2}[5 - (-2)] = \frac{7}{2}$$

$3 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$





# AP® Calculus AB 2014 Scoring Guidelines

## 2014 SCORING GUIDELINES

### Question 1

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and $t$ is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time $t$ for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to $A$ at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

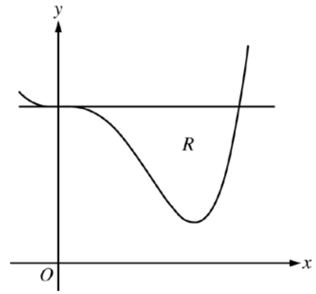
| | |
|---|---|
| (a) $\dfrac{A(30) - A(0)}{30 - 0} = -0.197$ (or $-0.196$) lbs/day | 1 : answer with units |
| (b) $A'(15) = -0.164$ (or $-0.163$)<br>The amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lbs/day at time $t = 15$ days. | $2 : \begin{cases} 1 : A'(15) \\ 1 : \text{interpretation} \end{cases}$ |
| (c) $A(t) = \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \Rightarrow t = 12.415$ (or 12.414) | $2 : \begin{cases} 1 : \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \\ 1 : \text{answer} \end{cases}$ |
| (d) $L(t) = A(30) + A'(30)\cdot(t - 30)$<br>$A'(30) = -0.055976$<br>$A(30) = 0.782928$<br>$L(t) = 0.5 \Rightarrow t = 35.054$ | $4 : \begin{cases} 2 : \text{expression for } L(t) \\ 1 : L(t) = 0.5 \\ 1 : \text{answer} \end{cases}$ |





## 2014 SCORING GUIDELINES

### Question 2




Let $R$ be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(a) Find the volume of the solid generated when $R$ is rotated about the horizontal line $y = -2$.

(b) Region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is an isosceles right triangle with a leg in $R$. Find the volume of the solid.

(c) The vertical line $x = k$ divides $R$ into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value $k$.

| | |
|---|---|
| (a) $f(x) = 4 \Rightarrow x = 0, 2.3$<br>Volume $= \pi\displaystyle\int_0^{2.3}\left[(4+2)^2 - (f(x)+2)^2\right]dx$<br>$= 98.868$ (or 98.867) | $4 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$ |
| (b) Volume $= \displaystyle\int_0^{2.3} \frac{1}{2}(4 - f(x))^2\,dx$<br>$= 3.574$ (or 3.573) | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (c) $\displaystyle\int_0^k (4 - f(x))\,dx = \int_k^{2.3} (4 - f(x))\,dx$ | $2 : \begin{cases} 1 : \text{area of one region} \\ 1 : \text{equation} \end{cases}$ |

## 2014 SCORING GUIDELINES

### Question 3

The function $f$ is defined on the closed interval $[-5, 4]$. The graph of $f$ consists of three line segments and is shown in the figure above.

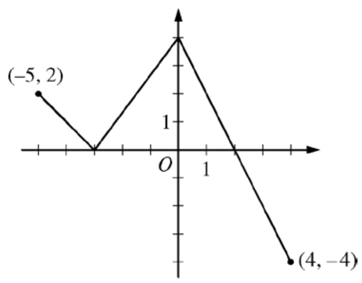
Let $g$ be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of $g$ both increasing and concave down? Give a reason for your answer.

(c) The function $h$ is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function $p$ is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of $p$ at the point where $x = -1$.



Graph of $f$

---

(a) $g(3) = \int_{-3}^{3} f(t)\,dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of $g$ is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \dfrac{5xg'(x) - g(x)5}{(5x)^2} = \dfrac{5xg'(x) - 5g(x)}{25x^2}$

$$h'(3) = \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2} = \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d) $p'(x) = f'\left(x^2 - x\right)(2x - 1)$

$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$

$3 : \begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$





## 2014 SCORING GUIDELINES

### Question 4

Train $A$ runs back and forth on an east-west section of railroad track. Train $A$'s velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time $t$ is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

| $t$ (minutes) | 0 | 2 | 5 | 8 | 12 |
|---|---|---|---|---|---|
| $v_A(t)$ (meters/minute) | 0 | 100 | 40 | –120 | –150 |

(a) Find the average acceleration of train $A$ over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train $A$'s velocity is $-100$ meters per minute at some time $t$ with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train $A$'s position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train $A$, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train $B$, travels north from the Origin Station. At time $t$ the velocity of train $B$ is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train $A$ and train $B$ is changing at time $t = 2$.

---

(a) average accel $= \dfrac{v_A(8) - v_A(2)}{8 - 2} = \dfrac{-120 - 100}{6} = -\dfrac{110}{3}$ m/min$^2$

1 : average acceleration

(b) $v_A$ is differentiable $\Rightarrow$ $v_A$ is continuous

$v_A(8) = -120 < -100 < 40 = v_A(5)$

Therefore, by the Intermediate Value Theorem, there is a time $t$, $5 < t < 8$, such that $v_A(t) = -100$.

$2 : \begin{cases} 1 : v_A(8) < -100 < v_A(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$

(c) $s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt$

$$\int_2^{12} v_A(t)\,dt \approx 3 \cdot \frac{100 + 40}{2} + 3 \cdot \frac{40 - 120}{2} + 4 \cdot \frac{-120 - 150}{2} = -450$$

$s_A(12) \approx 300 - 450 = -150$

The position of Train $A$ at time $t = 12$ minutes is approximately 150 meters west of Origin Station.

$3 : \begin{cases} 1 : \text{position expression} \\ 1 : \text{trapezoidal sum} \\ 1 : \text{position at time } t = 12 \end{cases}$

(d) Let $x$ be train $A$'s position, $y$ train $B$'s position, and $z$ the distance between train $A$ and train $B$.

$z^2 = x^2 + y^2 \Rightarrow 2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$

$x = 300,\ y = 400 \Rightarrow z = 500$

$v_B(2) = -20 + 120 + 25 = 125$

$500\dfrac{dz}{dt} = (300)(100) + (400)(125)$

$\dfrac{dz}{dt} = \dfrac{80000}{500} = 160$ meters per minute

$3 : \begin{cases} 2 : \text{implicit differentiation of distance relationship} \\ 1 : \text{answer} \end{cases}$

## 2014 SCORING GUIDELINES

### Question 5

| $x$ | $-2$ | $-2 < x < -1$ | $-1$ | $-1 < x < 1$ | $1$ | $1 < x < 3$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 12 | Positive | 8 | Positive | 2 | Positive | 7 |
| $f'(x)$ | $-5$ | Negative | 0 | Negative | 0 | Positive | $\frac{1}{2}$ |
| $g(x)$ | $-1$ | Negative | 0 | Positive | 3 | Positive | 1 |
| $g'(x)$ | 2 | Positive | $\frac{3}{2}$ | Positive | 0 | Negative | $-2$ |

The twice-differentiable functions $f$ and $g$ are defined for all real numbers $x$. Values of $f$, $f'$, $g$, and $g'$ for various values of $x$ are given in the table above.

(a) Find the $x$-coordinate of each relative minimum of $f$ on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value $c$, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function $h$ is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

---

(a) $x = 1$ is the only critical point at which $f'$ changes sign from negative to positive. Therefore, $f$ has a relative minimum at $x = 1$.

1 : answer with justification

(b) $f'$ is differentiable $\Rightarrow$ $f'$ is continuous on the interval $-1 \le x \le 1$

$$\frac{f'(1) - f'(-1)}{1 - (-1)} = \frac{0 - 0}{2} = 0$$

Therefore, by the Mean Value Theorem, there is at least one value $c$, $-1 < c < 1$, such that $f''(c) = 0$.

$2 : \begin{cases} 1 : f'(1) - f'(-1) = 0 \\ 1 : \text{explanation, using Mean Value Theorem} \end{cases}$

(c) $h'(x) = \dfrac{1}{f(x)} \cdot f'(x)$

$$h'(3) = \frac{1}{f(3)} \cdot f'(3) = \frac{1}{7} \cdot \frac{1}{2} = \frac{1}{14}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d) 
$$\begin{aligned} \int_{-2}^{3} f'(g(x))g'(x)\,dx &= \left[f(g(x))\right]_{x=-2}^{x=3} \\ &= f(g(3)) - f(g(-2)) \\ &= f(1) - f(-1) \\ &= 2 - 8 = -6 \end{aligned}$$

$3 : \begin{cases} 2 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$





## 2014 SCORING GUIDELINES

### Question 6

Consider the differential equation $\dfrac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function $f$ is defined for all real numbers.

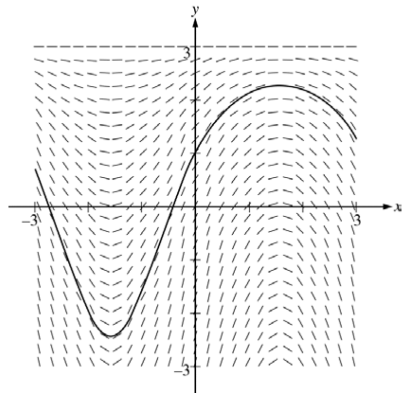
(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

---

(a)




1 : solution curve

(b) $\left.\dfrac{dy}{dx}\right|_{(x,\,y)=(0,\,1)} = 2\cos 0 = 2$

An equation for the tangent line is $y = 2x + 1$.

$f(0.2) \approx 2(0.2) + 1 = 1.4$

$2 : \begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(c) $\dfrac{dy}{dx} = (3 - y)\cos x$

$\displaystyle\int \frac{dy}{3 - y} = \int \cos x\,dx$

$-\ln|3 - y| = \sin x + C$

$-\ln 2 = \sin 0 + C \Rightarrow C = -\ln 2$

$-\ln|3 - y| = \sin x - \ln 2$

Because $y(0) = 1$, $y < 3$, so $|3 - y| = 3 - y$

$3 - y = 2e^{-\sin x}$

$y = 3 - 2e^{-\sin x}$

$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables