

AP[®] Calculus AB

2015 Scoring Guidelines

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2015 SCORING GUIDELINES

Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R , where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

- How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \leq t \leq 8$?
- Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.
- At what time t , $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.
- The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t) dt = 76.570$

2 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$

Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

2 : $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t , $0 \leq t \leq 8$, is $30 + \int_0^t [R(x) - D(x)] dx$.

3 : $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

$$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

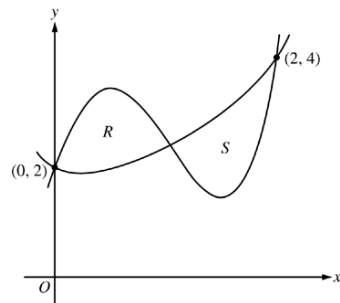
(d) $30 + \int_0^w [R(t) - D(t)] dt = 50$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

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Question 2

Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2-2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.



- (a) Find the sum of the areas of regions R and S .
- (b) Region S is the base of a solid whose cross sections perpendicular to the x -axis are squares. Find the volume of the solid.
- (c) Let h be the vertical distance between the graphs of f and g in region S . Find the rate at which h changes with respect to x when $x = 1.8$.

- (a) The graphs of $y = f(x)$ and $y = g(x)$ intersect in the first quadrant at the points $(0, 2)$, $(2, 4)$, and $(A, B) = (1.032832, 2.401108)$.

$$\begin{aligned}\text{Area} &= \int_0^A [g(x) - f(x)] dx + \int_A^2 [f(x) - g(x)] dx \\ &= 0.997427 + 1.006919 = 2.004\end{aligned}$$

- (b) Volume $= \int_A^2 [f(x) - g(x)]^2 dx = 1.283$

- (c) $h(x) = f(x) - g(x)$
 $h'(x) = f'(x) - g'(x)$
 $h'(1.8) = f'(1.8) - g'(1.8) = -3.812$ (or -3.811)

4 : $\begin{cases} 1 : \text{limits} \\ 2 : \text{integrands} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{considers } h' \\ 1 : \text{answer} \end{cases}$

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Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	-220	150

Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function v . Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

- (a) Use the data in the table to estimate the value of $v'(16)$.

- (b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)| dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)| dt$ using a right Riemann sum with the four subintervals indicated in the table.

- (c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

- (d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

- (a) $v'(16) \approx \frac{240 - 200}{20 - 12} = 5$ meters/min²

1 : approximation

- (b) $\int_0^{40} |v(t)| dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \leq t \leq 40$ minutes.

3 : $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned}\int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters}\end{aligned}$$

- (c) Bob's acceleration is $B'(t) = 3t^2 - 12t$.

$$B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$$

2 : $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

- (d) Avg vel $= \frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$
 $= \frac{1}{10} \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10}$
 $= \frac{1}{10} \left[\frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min}$

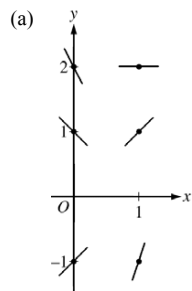
3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

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Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.
- (d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.



- 2 : $\begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

- 2 : $\begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c) $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

- 2 : $\begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

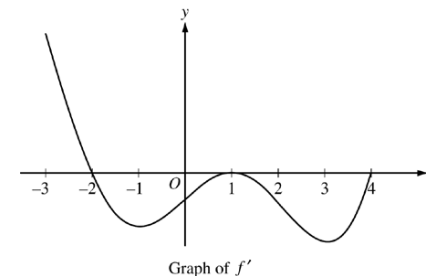
$$b = -m \Rightarrow b = -2$$

- 3 : $\begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

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Question 5

The figure above shows the graph of f' , the derivative of a twice-differentiable function f , on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x -axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.



- (a) Find all x -coordinates at which f has a relative maximum. Give a reason for your answer.
- (b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.
- (c) Find the x -coordinates of all points of inflection for the graph of f . Give a reason for your answer.
- (d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

- (a) $f'(x) = 0$ at $x = -2$, $x = 1$, and $x = 4$.
 $f'(x)$ changes from positive to negative at $x = -2$.
Therefore, f has a relative maximum at $x = -2$.

- 2 : $\begin{cases} 1 : \text{identifies } x = -2 \\ 1 : \text{answer with reason} \end{cases}$

- (b) The graph of f is concave down and decreasing on the intervals $-2 < x < -1$ and $1 < x < 3$ because f' is decreasing and negative on these intervals.

- 2 : $\begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

- (c) The graph of f has a point of inflection at $x = -1$ and $x = 3$ because f' changes from decreasing to increasing at these points.

- 2 : $\begin{cases} 1 : \text{identifies } x = -1, 1, \text{ and } 3 \\ 1 : \text{reason} \end{cases}$

The graph of f has a point of inflection at $x = 1$ because f' changes from increasing to decreasing at this point.

(d) $f(x) = 3 + \int_1^x f'(t) dt$

$$f(4) = 3 + \int_1^4 f'(t) dt = 3 + (-12) = -9$$

$$f(-2) = 3 + \int_1^{-2} f'(t) dt = 3 - \int_{-2}^1 f'(t) dt = 3 - (-9) = 12$$

- 3 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{expression for } f(x) \\ 1 : f(4) \text{ and } f(-2) \end{cases}$

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Question 6

Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

- (a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.
- (b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.
- (c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

$$(a) \quad \left. \frac{dy}{dx} \right|_{(x,y)=(-1,1)} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{4}$$

An equation for the tangent line is $y = \frac{1}{4}(x + 1) + 1$.

$$(b) \quad 3y^2 - x = 0 \Rightarrow x = 3y^2$$

$$\text{So, } y^3 - xy = 2 \Rightarrow y^3 - (3y^2)(y) = 2 \Rightarrow y = -1$$

$$(-1)^3 - x(-1) = 2 \Rightarrow x = 3$$

The tangent line to the curve is vertical at the point $(3, -1)$.

$$(c) \quad \frac{d^2y}{dx^2} = \frac{(3y^2 - x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$$

$$\left. \frac{d^2y}{dx^2} \right|_{(x,y)=(-1,1)} = \frac{(3 \cdot 1^2 - (-1)) \cdot \frac{1}{4} - 1 \cdot \left(6 \cdot 1 \cdot \frac{1}{4} - 1\right)}{(3 \cdot 1^2 - (-1))^2}$$

$$= \frac{1 - \frac{1}{2}}{16} = \frac{1}{32}$$

$$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{equation for tangent line} \end{cases}$$

$$3 : \begin{cases} 1 : \text{sets } 3y^2 - x = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{coordinates} \end{cases}$$

$$4 : \begin{cases} 2 : \text{implicit differentiation} \\ 1 : \text{substitution for } \frac{dy}{dx} \\ 1 : \text{answer} \end{cases}$$