Week 20

AP Calculus AB

Homework bundle for this week

本周作业合集

exercises.lol

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AP Calculus AB Name: ____________ Score: ______

1. A tap runs at 3 L/min for 4 min. The area under the rate graph (accumulated volume) is...

   ____

2. For $f(x) = x^2$ on $[0, 2]$ with 2 right rectangles (width 1): heights $f(1) = 1$, $f(2) = 4$. Find the right Riemann sum.

   ____

3. For n equal strips on $[a, b]$, the width Δx is...

   ☐ $\dfrac{b-a}{n}$

   ☐ $b - a$

   ☐ $\dfrac{n}{b-a}$

   ☐ $n(b - a)$

4. By FTC Part 1, $\dfrac{d}{dx} \int_a^x f(t)\, dt =$

   ☐ $F(x) - F(a)$

   ☐ $f(x)$

   ☐ $f'(x)$

   ☐ $f(a)$

5. For $g(x) = \int_a^x f(t)\, dt$, g is increasing exactly where...

   ☐ $f(x) > 0$

   ☐ $f(x) < 0$

   ☐ $f'(x) > 0$

   ☐ $g(x) > 0$

6. Given $\int_0^4 f\, dx = 10$ and $\int_0^4 g\, dx = 3$, find $\int_0^4 (2f - g)\, dx$.

   ____

7. FTC Part 2 says $\int_a^b f\, dx =$

   ☐ $f'(b) - f'(a)$

   ☐ $F(b) - F(a)$ where $F' = f$

   ☐ $F(a) - F(b)$

   ☐ $f(b) - f(a)$

8. What is $\int x^3\,dx$?

- ☐ $3x^2 + C$
- ☐ $\dfrac{x^4}{4} + C$
- ☐ $x^4 + C$
- ☐ $\dfrac{x^2}{2} + C$

9. Where a rate graph dips below the axis, that area subtracts from the net accumulated change.

☐ True ☐ False

10. Same setup, but a **left** Riemann sum: heights $f(0) = 0$, $f(1) = 1$. Find it.

____

AP Calculus AB

1. **12**

   Rectangle area $= 3 \times 4 = 12$ L.

2. **5**

   $1 \cdot 1 + 4 \cdot 1 = 5.$

3. $\dfrac{b-a}{n}$

   The interval width divided by the number of strips.

4. $f(x)$

   The derivative of the accumulation function is the integrand $f(x)$.

5. $f(x) > 0$

   $g' = f$, so g increases where $f > 0$.

6. **17**

   $2(10) - 3 = 17.$

7. $F(b) - F(a)$ **where** $F' = f$

   Evaluate an antiderivative at upper minus lower limit.

8. $\dfrac{x^4}{4} + C$

   Add one to the exponent, divide by it: $\frac{x^4}{4} + C$.

9. **True**

   Below-axis area is negative (signed area).

10. **1**

    $0 \cdot 1 + 1 \cdot 1 = 1.$

AP CALCULUS AB • EXERCISE SHEET

6.1 Exploring Accumulations of Change

Name: ______________ Class: __________ Date: ____________ **Total: 10 marks**

KEY WORDS accumulated change 累积变化

Objective

Build the skills to answer exam questions on **accumulation of change** — reading a total change as the area under a rate graph.

You must be able to:

- interpret the **area** under a rate-of-change graph as an **accumulated change** 累积变化
- treat area **above** the axis as positive and area **below** as negative
- attach correct **units** (rate unit × time unit) to an accumulation

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Area under a rate graph is a total

If water flows in at a **rate** $R(t)$ litres per minute, the total water added between $t = 0$ and $t = 5$ is the **area** under the R-vs-t graph over $[0, 5]$. A rate graph that is a rectangle of height 3 L/min over 5 min accumulates $3 \times 5 = 15$ L.

■ Splitting into simple shapes

If the rate graph is made of straight lines, find the area with triangles and rectangles. A rate rising linearly from 0 to 4 L/min over 6 min is a triangle: area $= \frac{1}{2}(6)(4) = 12$ L.

■ Signed area

When a rate goes **negative** (water leaving), that area counts as a **loss**. Over a stretch where the graph is $+8$ area then -3 area, the net accumulated change is $8 - 3 = +5$.

■ Units

The accumulation's unit is (rate unit) × (time unit). A rate in L/min over minutes accumulates **litres**; a velocity in m/s over seconds accumulates **metres**.

2 Practice

Now apply the methods above.

2.1 A tap runs at a constant 2.5 L/min for 8 minutes. How much water flows out? [1]

2.2 A rate graph is a triangle: the rate rises linearly from 0 to 10 units/s over 4 s. Find the accumulated change. [2]

2.3 Over one interval a quantity gains an area of 12 and over the next it loses an area of 5. State the net accumulated change. [1]

3 Exam-style questions

3.1 The area under a velocity-vs-time graph represents [1]

- **A** acceleration
- **B** displacement
- **C** speed
- **D** force

3.2 A machine consumes power at a rate $P(t)$ kilowatts, shown as a graph made of straight segments: a rectangle of height 2 over $0 \le t \le 3$ hours, then a triangle rising from 0 to 4 over $3 \le t \le 5$ hours.

(a) Find the energy used over $0 \le t \le 3$. [1]

(b) Find the energy used over $3 \le t \le 5$. [2]

(c) State the total energy used, with units. [1]

3.3 Sand is added to a pile at a rate that gives an accumulated area of 30 (kg) over the first hour, then removed at a rate giving an area of 8 (kg) over the next hour. State the net change in the pile's mass. [1]

Solutions

2.1 $2.5 \times 8 = 20$ L.

2.2 Triangle area $= \frac{1}{2}(4)(10) = 20$ units.

2.3 $12 - 5 = 7$.

3.1 **B** — the area under a velocity–time graph is displacement.

3.2 (a) $2 \times 3 = 6$ kWh. (b) triangle $\frac{1}{2}(2)(4) = 4$ kWh. (c) $6 + 4 = 10$ kWh.

3.3 $30 - 8 = 22$ kg.

CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

AP CALCULUS AB • EXERCISE SHEET

6.2 Approximating Areas with Riemann Sums

Name: ______________ Class: __________ Date: ____________ **Total: 12 marks**

KEY WORDS riemann sum 黎曼和 • trapezoidal 梯形 • trapezoidal sum 梯形法 • midpoint 中点

Objective

Build the skills to answer exam questions on **Riemann sums** — estimating the area under a curve with rectangles.

You must be able to:

- compute **left**, **right**, and **midpoint** 中点 Riemann sums from a function or a table
- compute a **trapezoidal** 梯形 estimate
- decide whether an estimate **over-** or **under-estimates** using increasing/decreasing or concavity

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Left and right sums

Estimate $\int_0^6 f\,dx$ with 3 equal subintervals of width $\Delta x = 2$, using the table:

x	**0**	**2**	**4**	**6**
$f(x)$	1	4	9	16

- **Left** sum uses the left endpoints: $2(1+4+9) = 28$.
- **Right** sum uses the right endpoints: $2(4+9+16) = 58$.

Midpoint sum

The **midpoint** sum uses the value at the middle of each subinterval. With $f(x) = x^2$ on $[0, 6]$, $n = 3$, $\Delta x = 2$, midpoints $1, 3, 5$: $2\big(f(1)+f(3)+f(5)\big) = 2(1+9+25) = 70$.

■ **Trapezoidal estimate**

A **trapezoidal** sum averages each pair of endpoint heights: $\Delta x\left[\frac{f_0}{2}+f_1+f_2+\frac{f_3}{2}\right]$. For the table: $2\left[\frac{1}{2}+4+9+\frac{16}{2}\right]=2(21.5)=43$. It equals the average of the left and right sums.

■ **Over- or under-estimate**

For an **increasing** function, a left sum **under**-estimates and a right sum **over**-estimates. For a **concave-up** function, the trapezoidal sum **over**-estimates and the midpoint sum **under**-estimates.

2 Practice

Now apply the methods above.

Use the table for 2.1–2.3:

x	**0**	**3**	**6**	**9**
$g(x)$	2	5	6	10

2.1 Find the **left** Riemann sum for $\int_0^9 g\,dx$ with the three subintervals shown. [2]

2.2 Find the **right** Riemann sum for the same integral. [2]

2.3 Find the **trapezoidal** estimate. [2]

3 Exam-style questions

3.1 For an increasing function, a **left** Riemann sum gives an [1]

- **A** over-estimate
- **B** under-estimate
- **C** exact value

- **D** value that depends on concavity

3.2 A car's speed $v(t)$ in m/s is recorded:

t **(s)**	**0**	**4**	**8**	**12**
$v(t)$	0	12	20	24

(a) Estimate the distance travelled over $0 \le t \le 12$ using a **right** Riemann sum with three subintervals. [2]

(b) Is this an over- or under-estimate of the true distance? Give a reason. [1]

3.3 The function f is concave up on $[0, 4]$. A student estimates $\int_0^4 f\,dx$ with a trapezoidal sum. State, with a reason, whether the estimate is too large or too small. [2]

Solutions

2.1 $\Delta x = 3$; left $= 3(2+5+6) = 39$.

2.2 right $= 3(5+6+10) = 63$.

2.3 $3\left[\frac{2}{2}+5+6+\frac{10}{2}\right] = 3(17) = 51$. (Equivalently the average of 39 and 63.)

3.1 **B** — for an increasing function the left sum under-estimates.

3.2 (a) $4(12+20+24) = 224$ m. (b) Over-estimate — v is increasing, so right endpoints are the largest values on each subinterval.

3.3 Too large — for a concave-up curve each trapezoid lies **above** the curve, so the trapezoidal sum over-estimates.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0,10]$, $[10,20]$, and $[20,30]$ to approximate $\int_0^{30} M(t)dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t+15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

   (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t)\,dt$.

   (b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

   (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

   Show the setup for your calculations.

   (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\, dt$.

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

(a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r)\, dr$. Approximate the value of $2\pi \int_0^4 r f(r)\, dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	–29	55	48

2. The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

(a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour.

(b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\,dt$.

(c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.

(d) At time $t = 0$, particle Q is at position $x = -90$. Using the result from part (b) and the function v_Q from part (c), approximate the distance between particles P and Q at time $t = 2.8$.

END OF PART A OF SECTION II

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

   (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

   (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

   (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

   (d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

   (a) Use the data in the table to estimate the value of $v'(16)$.

   (b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

   (c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

   (d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	–120	–150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

6.3 Riemann Sums and Integral Notation

Name: ______________ Class: __________ Date: ____________ **Total: 11 marks**

KEY WORDS definite integral 定积分 • riemann sum 黎曼和 • antiderivative 原函数

Objective

Build the skills to answer exam questions on **Riemann sums and definite-integral notation** — connecting a limit of sums to $\int_a^b f\,dx$.

You must be able to:

- write a Riemann sum with **summation notation** $\sum$
- recognise the **definite integral** 定积分 as the limit of Riemann sums as $n \to \infty$
- read the meaning of the parts of $\displaystyle\int_a^b f(x)\,dx$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Summation notation for a sum

A right Riemann sum with n equal pieces on $[a, b]$ has width $\Delta x = \dfrac{b-a}{n}$ and sample points $x_k = a + k\,\Delta x$:

$$\sum_{k=1}^{n} f(x_k)\,\Delta x.$$

The definite integral as a limit

Letting the rectangles become infinitely thin gives the exact area:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

The integral **is** this limit — a sum of infinitely many infinitely thin pieces.

■ **Reading the notation**

In $\int_a^b f(x)\,dx$: a and b are the **limits of integration**, $f(x)$ is the **integrand**, and dx marks the variable and the "width" of each piece. The result is a **number** (a signed area), not a function.

■ **From a limit back to an integral**

A limit like $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{1+\frac{3k}{n}}\cdot\frac{3}{n}$ is a right sum with $\Delta x=\frac{3}{n}$ on $[1,4]$, so it equals $\int_1^4 \sqrt{x}\,dx$.

2 Practice

Now apply the methods above.

2.1 For $\int_2^{10} f\,dx$ estimated with $n=4$ equal subintervals, state Δx. [1]

2.2 Write, in summation notation, a right Riemann sum for $\int_0^2 f\,dx$ with n equal subintervals. [2]

2.3 In $\int_1^5 (3x+2)\,dx$, name the integrand and the limits of integration. [2]

3 Exam-style questions

3.1 The definite integral $\int_a^b f\,dx$ is defined as [1]

- **A** an antiderivative of f
- **B** the limit of Riemann sums as $n\to\infty$
- **C** the slope of f
- **D** the average of $f(a)$ and $f(b)$

3.2 Consider $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{4k}{n}\right)^2 \cdot \frac{4}{n}$.

(a) State the width Δx and the interval $[a, b]$. [2]

(b) Write the limit as a definite integral. [2]

3.3 A definite integral $\int_0^6 v(t)\,dt$ evaluates to a negative number. Explain what a **negative** value tells you about the signed area, in one sentence. [1]

Solutions

2.1 $\Delta x = \dfrac{10-2}{4} = 2$.

2.2 $\Delta x = \dfrac{2}{n}$, $x_k = \dfrac{2k}{n}$; sum $= \displaystyle\sum_{k=1}^{n} f\left(\tfrac{2k}{n}\right) \cdot \tfrac{2}{n}$.

2.3 Integrand $3x+2$; limits of integration 1 and 5.

3.1 **B** — the definite integral is the limit of Riemann sums.

3.2 (a) $\Delta x = \dfrac{4}{n}$, so $b-a=4$; the sample $2+\frac{4k}{n}$ starts at 2, so $[a,b]=[2,6]$. (b) $\displaystyle\int_2^6 x^2\,dx$.

3.3 More signed area lies **below** the axis than above it over $[0,6]$.

1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

6.4 The Fundamental Theorem of Calculus and Accumulation Functions

Name: ______________ Class: __________ Date: ____________ **Total: 11 marks**

KEY WORDS accumulation function 累积函数 • fundamental theorem of calculus 微积分基本定理

Objective

Build the skills to answer exam questions on the **Fundamental Theorem of Calculus and accumulation functions** — differentiating an integral with a variable limit.

You must be able to:

- use $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ (the **accumulation function 累积函数**)
- apply the **chain rule** when the upper limit is a function of x
- evaluate an accumulation function $g(x) = \displaystyle\int_a^x f(t)\,dt$ at given points

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The Fundamental Theorem (derivative form)

If $g(x) = \displaystyle\int_a^x f(t)\,dt$, then $g'(x) = f(x)$. Differentiating "undoes" the integral. For $g(x) = \displaystyle\int_2^x \cos(t^2)\,dt$, we get $g'(x) = \cos(x^2)$ — no integration needed.

■ Chain rule with a variable upper limit

If the upper limit is $u(x)$, then

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f\big(u(x)\big)\cdot u'(x).$$

For $\displaystyle\int_0^{x^2} \sin t\,dt$: the answer is $\sin(x^2)\cdot 2x$.

■ **Evaluating an accumulation function**

$g(x) = \int_1^x (2t)\,dt$. A value like $g(3) = \int_1^3 2t\,dt = [t^2]_1^3 = 9 - 1 = 8$, and $g(1) = 0$ (zero-width interval).

■ **Sign of the accumulation**

g **increases** where $f > 0$ and **decreases** where $f < 0$, because $g' = f$. Reading a graph of f tells you where g rises and falls.

2 Practice

Now apply the methods above.

2.1 If $g(x) = \int_0^x (t^3 + 1)\,dt$, find $g'(x)$. [1]

2.2 Find $\frac{d}{dx} \int_1^x \sqrt{t}\,dt$. [1]

2.3 Find $\frac{d}{dx} \int_0^{x^3} \cos t\,dt$. [2]

3 Exam-style questions

3.1 If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) =$ [1]

- **A** $f'(x)$
- **B** $f(x)$
- **C** $f(x) - f(a)$
- **D** $F(x) - F(a)$

3.2 Let $g(x) = \int_2^x f(t)\,dt$, where f is continuous.

(a) Find $g(2)$. [1]

(b) Given $f(5) = 7$, find $g'(5)$. [1]

(c) If $f(t) > 0$ for all $t > 2$, state whether g is increasing or decreasing for $x > 2$, with a reason. [2]

3.3 Let $h(x) = \displaystyle\int_0^{2x} e^{-t^2}\,dt$. Find $h'(x)$. [2]

Solutions

2.1 $g'(x) = x^3 + 1$.

2.2 $\sqrt{x}$.

2.3 $\cos(x^3) \cdot 3x^2$.

3.1 **B** — by the Fundamental Theorem, $g'(x) = f(x)$.

3.2 (a) $g(2) = 0$ (zero-width interval). (b) $g'(5) = f(5) = 7$. (c) Increasing — $g'(x) = f(x) > 0$ for $x > 2$.

3.3 $h'(x) = e^{-(2x)^2} \cdot 2 = 2e^{-4x^2}$.

6. The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	0	2	3	6
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Part A

Find $\lim_{x \to 2} \frac{f(x)}{x}$, or state that the limit does not exist.

Part B

Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

Part C

Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

Part D

Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\,dt$.

i. Find $k'(x)$.

ii. Find $k''(3)$. Show the work that leads to your answer.

STOP

END OF EXAM

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6 \arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

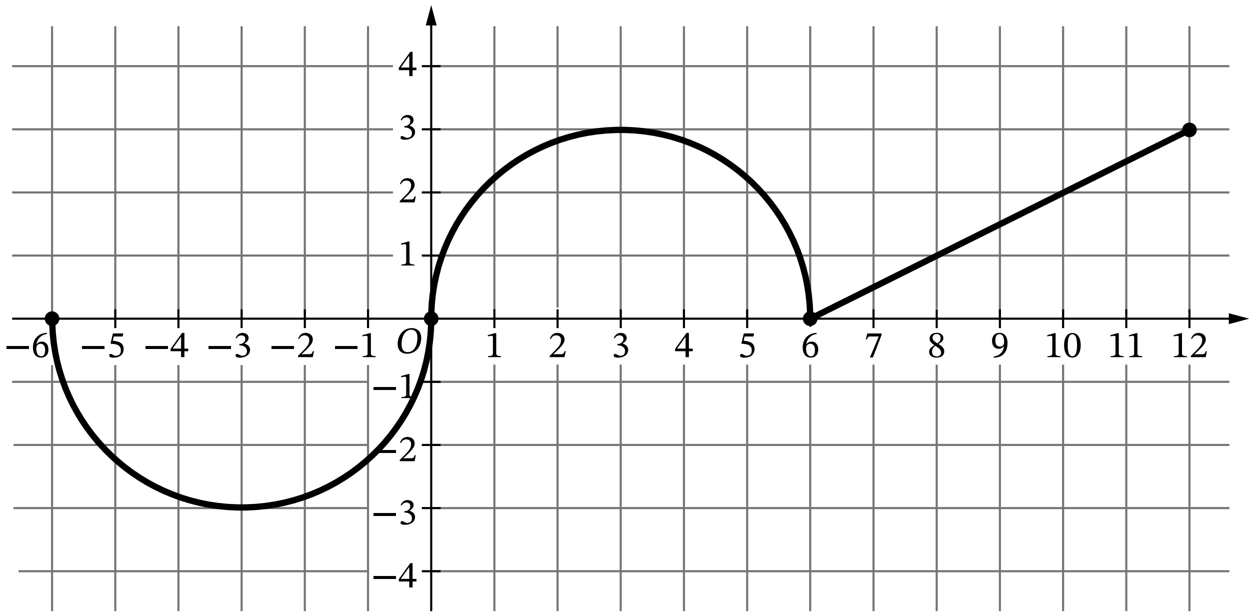
A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

4. The continuous function f is defined on the closed interval $-6 \le x \le 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.



Graph of f

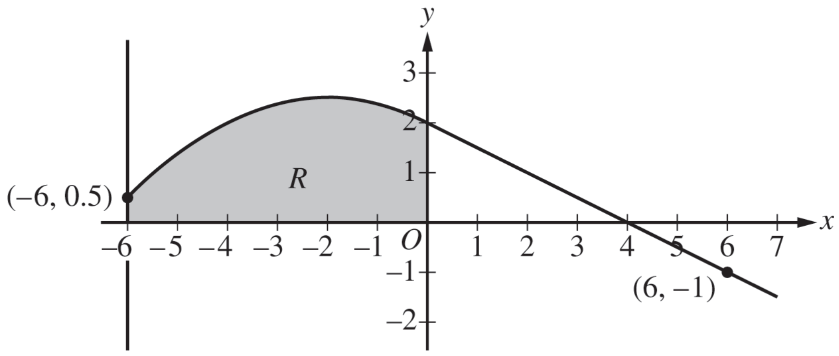
Let g be the function defined by $g(x) = \int_6^x f(t)\,dt$.

A. Find $g'(8)$. Give a reason for your answer.

B. Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.

C. Find $g(12)$ and $g(0)$. Label your answers.

D. Find the value of x at which g attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.



Graph of f

4. The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\, dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\, dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

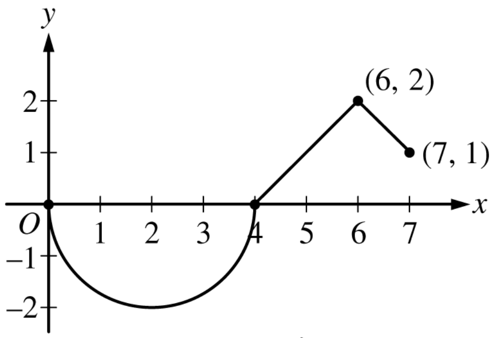
   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f'

3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

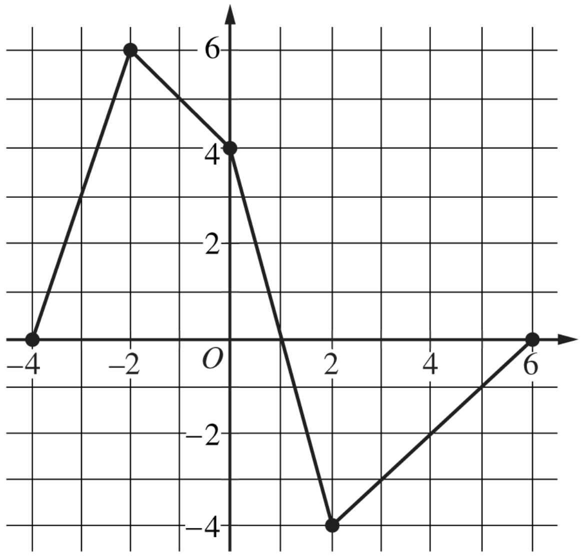
(a) Find $f(0)$ and $f(5)$.

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\, dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

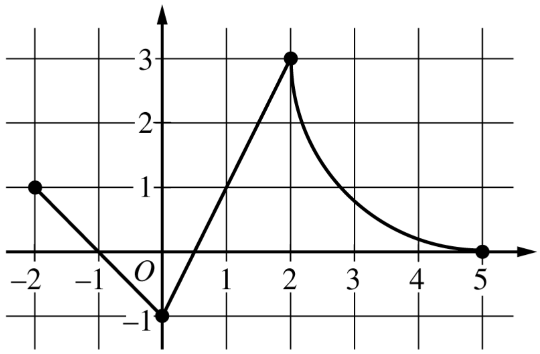
CALCULUS AB

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of problems—2

A graphing calculator is required for these problems.

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

(c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

6.5 Behavior of Accumulation Functions

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS local extrema 极值 • accumulation function 累积函数

Objective

Build the skills to answer exam questions on the **behavior of accumulation functions** — reading features of $g(x) = \int_a^x f(t)\,dt$ from a graph of f.

You must be able to:

- decide where g is **increasing/decreasing** from the sign of f
- locate **local extrema 极值** of g where f crosses zero (sign change)
- decide **concavity** of g from whether f is increasing or decreasing

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ g rises where f is positive

Because $g'(x) = f(x)$, the accumulation function g **increases** on intervals where the graph of f is **above** the axis and **decreases** where f is **below** the axis.

■ Extrema of g at zeros of f

g has a **local maximum** where f changes from $+$ to $-$, and a **local minimum** where f changes from $-$ to $+$. A zero of f where the sign does **not** change is not an extremum of g.

■ Concavity of g from the slope of f

$g''(x) = f'(x)$, so g is **concave up** where f is **increasing** and **concave down** where f is **decreasing**. A point where f has a local max or min is a **point of inflection** of g.

■ Comparing values of g

To compare $g(3)$ and $g(0)$, look at the net signed area of f between them: $g(3)-g(0) = \int_0^3 f\,dt$. More area above the axis than below makes $g(3) > g(0)$.

2 Practice

Now apply the methods above. Let $g(x) = \displaystyle\int_0^x f(t)\,dt$, where f is continuous.

2.1 On an interval where $f(t) < 0$, state whether g is increasing or decreasing. [1]

2.2 At a point where f changes from positive to negative, what feature does g have? [1]

2.3 On an interval where f is increasing, state the concavity of g. [1]

3 Exam-style questions

3.1 $g(x) = \int_0^x f(t)\,dt$. The graph of f crosses from below the axis to above it at $x = 4$. At $x = 4$, g has a [1]

- **A** local maximum
- **B** local minimum
- **C** point of inflection
- **D** vertical asymptote

3.2 The graph of f consists of straight segments: $f > 0$ on $(0, 3)$, $f < 0$ on $(3, 7)$, and $f(3) = 0$. Let $g(x) = \int_0^x f(t)\,dt$.

(a) On what interval is g increasing? [1]

(b) At what x does g attain a local maximum? Justify. [2]

3.3 For $g(x) = \int_0^x f(t)\,dt$, the graph of f is increasing on $(1, 5)$. State, with a reason, the concavity of g on $(1, 5)$. [2]

Solutions

2.1 Decreasing — $g'(x) = f(x) < 0$.

2.2 A local maximum.

2.3 Concave up — $g''(x) = f'(x) > 0$.

3.1 B — f changes $-$ to $+$, so g' changes sign the same way and g has a local minimum.

3.2 (a) g increases on $(0, 3)$ where $f > 0$. (b) At $x = 3$, because f (which is g') changes from positive to negative there.

3.3 Concave up — $g''(x) = f'(x) > 0$ since f is increasing.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0, 10]$, $[10, 20]$, and $[20, 30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

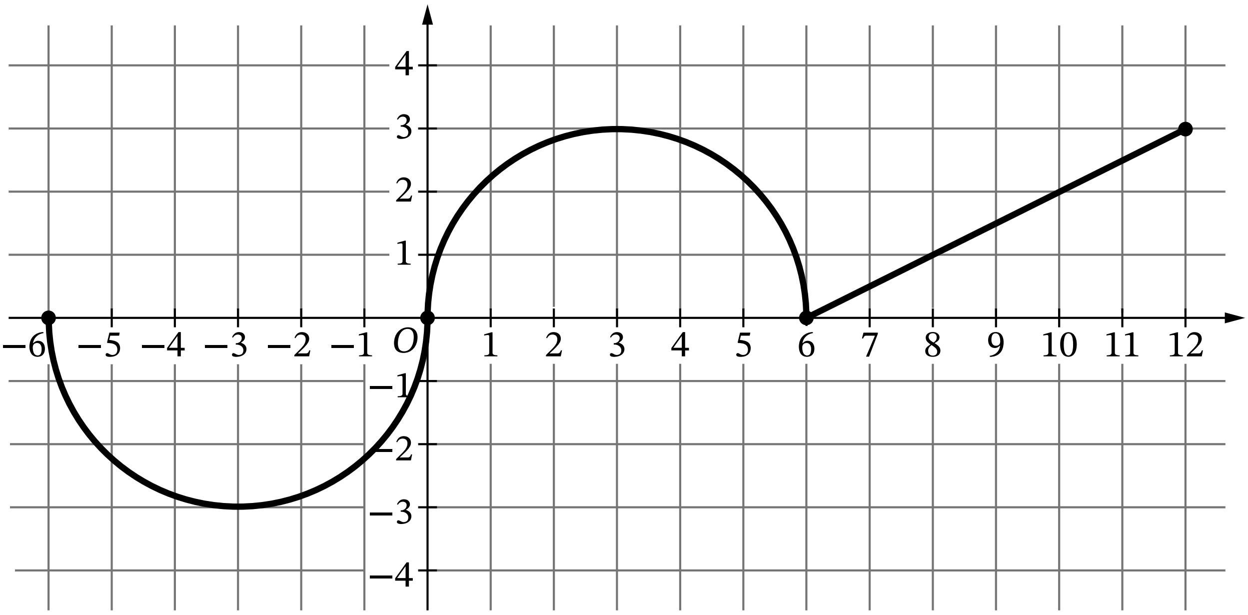
$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t + 15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

4. The continuous function f is defined on the closed interval $-6 \le x \le 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_6^x f(t)\,dt$.

A. Find $g'(8)$. Give a reason for your answer.

B. Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.

C. Find $g(12)$ and $g(0)$. Label your answers.

D. Find the value of x at which g attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\, dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t)\, dt$.

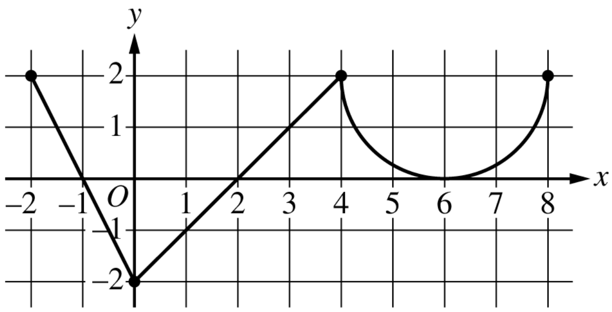
(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f'

4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

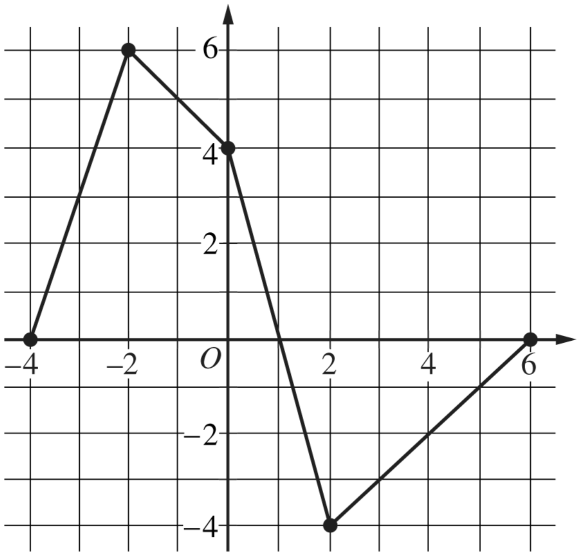
(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim\limits_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\dfrac{t}{100}\right)^3\left(1 - \dfrac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

(d) For $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

2. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

(b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

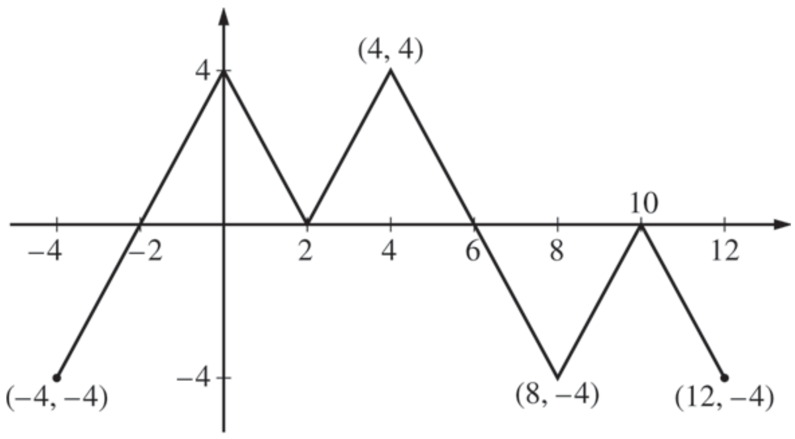
(c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

(d) How many pounds of bananas are on the display table at time $t = 8$?

END OF PART A OF SECTION II

CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

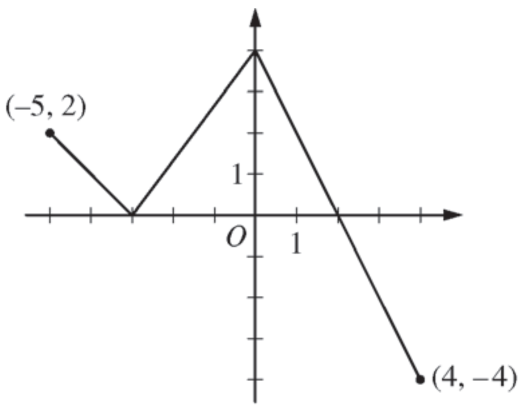
(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

   (a) Find $g(3)$.

   (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

   (c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

   (d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

6.6 Properties of Definite Integrals

Name: ______________ Class: __________ Date: ____________ **Total: 10 marks**

KEY WORDS definite integral 定积分

Objective

Build the skills to answer exam questions on the **properties of definite integrals** — combining and manipulating integral values.

You must be able to:

- use $\displaystyle\int_a^a f = 0$ and $\displaystyle\int_b^a f = -\int_a^b f$
- **split** an integral over adjacent intervals: $\displaystyle\int_a^c f = \int_a^b f + \int_b^c f$
- factor out constants and add integrands: $\displaystyle\int_a^b (cf + g) = c\int_a^b f + \int_a^b g$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Zero-width and reversing

$\displaystyle\int_a^a f\,dx = 0$ (no width). Reversing the limits flips the sign: $\displaystyle\int_b^a f\,dx = -\int_a^b f\,dx$.

■ Adding adjacent intervals

Integrals join end-to-end: $\displaystyle\int_1^4 f = \int_1^3 f + \int_3^4 f$. If $\int_1^3 f = 5$ and $\int_1^4 f = 9$, then $\int_3^4 f = 9 - 5 = 4$.

■ Linearity

Constants come out and sums split:

$$\int_a^b \big(3f(x) + 2g(x)\big)\,dx = 3\int_a^b f + 2\int_a^b g.$$

If $\int_0^2 f = 4$ and $\int_0^2 g = 1$, then $\int_0^2 (3f + 2g) = 3(4) + 2(1) = 14$.

■ **Combining with a reversal**

Given $\int_2^5 f = 10$, find $\int_5^2 3f$. Reverse and factor: $\int_5^2 3f = 3\int_5^2 f = 3(-10) = -30$.

2 Practice

Now apply the methods above. You are given $\displaystyle\int_0^3 f = 6$, $\displaystyle\int_3^7 f = -2$, and $\displaystyle\int_0^3 g = 5$.

2.1 Find $\displaystyle\int_0^7 f\,dx$. [1]

2.2 Find $\displaystyle\int_3^0 f\,dx$. [1]

2.3 Find $\displaystyle\int_0^3 (2f+g)\,dx$. [2]

3 Exam-style questions

3.1 $\int_5^5 f(x)\,dx =$ [1]

- **A** $f(5)$
- **B** 1
- **C** 0
- **D** undefined

3.2 You are given $\int_1^6 h = 12$ and $\int_1^4 h = 7$.

(a) Find $\int_4^6 h\,dx$. [2]

(b) Find $\int_6^1 h\,dx$. [1]

3.3 Given $\int_0^2 p = 8$ and $\int_0^2 q = -3$, find $\int_0^2 (5p - 4q)\,dx$. [2]

Solutions

2.1 $\int_0^7 f = \int_0^3 f + \int_3^7 f = 6 + (-2) = 4.$

2.2 $\int_3^0 f = -\int_0^3 f = -6.$

2.3 $2(6) + 5 = 17.$

3.1 **C** — a zero-width interval gives 0.

3.2 (a) $\int_4^6 h = \int_1^6 h - \int_1^4 h = 12 - 7 = 5.$ (b) $\int_6^1 h = -12.$

3.3 $5(8) - 4(-3) = 40 + 12 = 52.$

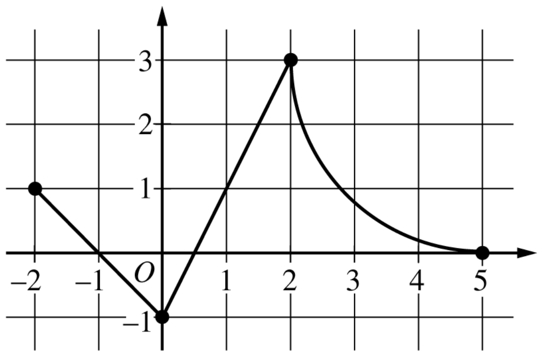
CALCULUS AB

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\, dx = 7$, find the value of $\int_{-6}^{-2} f(x)\, dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\, dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\, dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

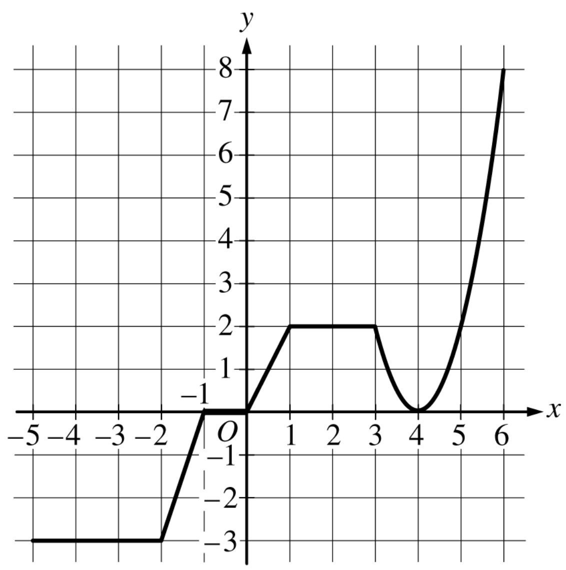
CALCULUS AB

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x - 4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

AP CALCULUS AB • EXERCISE SHEET

6.7 The Fundamental Theorem and Evaluating Integrals

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

KEY WORDS net change 净变化 • antiderivative 原函数 • fundamental theorem of calculus 微积分基本定理 • definite integral 定积分

Objective

Build the skills to answer exam questions on the **Fundamental Theorem of Calculus (evaluation form)** — using an antiderivative to evaluate a definite integral.

You must be able to:

- apply $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, where $F' = f$
- evaluate integrals of powers, exponentials, and basic trig functions
- interpret $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$ as a **net change 净变化**

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Evaluating with an antiderivative

$\displaystyle\int_1^3 2x\,dx$: an antiderivative of $2x$ is x^2, so

$$\int_1^3 2x\,dx = \left[x^2\right]_1^3 = 3^2 - 1^2 = 8.$$

■ A power and a constant

$\displaystyle\int_0^2 (3x^2+1)\,dx = \left[x^3 + x\right]_0^2 = (8+2) - 0 = 10$. Antidifferentiate term by term, then subtract the values at the limits.

■ **Net change from a rate**

If f' is a rate of change, then $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$ is the **net change** in f. If a tank's level changes at rate $r(t)$ and $\int_0^{10} r\,dt = -4$, the level **fell** by 4 over those 10 units.

■ **Trig and exponential**

$\displaystyle\int_0^{\pi} \sin x\,dx = \big[-\cos x\big]_0^{\pi} = -\cos\pi + \cos 0 = 1+1 = 2$, and $\displaystyle\int_0^1 e^x\,dx = \big[e^x\big]_0^1 = e-1$.

2 Practice

Now apply the methods above.

2.1 Evaluate $\displaystyle\int_0^4 x\,dx$. [2]

2.2 Evaluate $\displaystyle\int_1^2 (4x^3)\,dx$. [2]

2.3 Evaluate $\displaystyle\int_0^{\pi/2} \cos x\,dx$. [2]

3 Exam-style questions

3.1 $\displaystyle\int_a^b f'(x)\,dx =$ [1]

- **A** $f'(b) - f'(a)$
- **B** $f(b) - f(a)$
- **C** $f(b) + f(a)$
- **D** $\frac{1}{2}(f(a) + f(b))$

3.2 A particle's velocity is $v(t) = 3t^2 - 4$ m/s.

(a) Evaluate $\displaystyle\int_0^2 v(t)\,dt$. [2]

(b) State what this value represents physically. [1]

3.3 Water enters a tank at rate $r(t) = 6t$ L/min. Find the total volume added between $t = 1$ and $t = 4$ minutes. [3]

Solutions

2.1 $\left[\frac{1}{2}x^2\right]_0^4 = 8 - 0 = 8.$

2.2 $\left[x^4\right]_1^2 = 16 - 1 = 15.$

2.3 $\left[\sin x\right]_0^{\pi/2} = 1 - 0 = 1.$

3.1 **B** — by the FTC, $\int_a^b f' = f(b) - f(a)$.

3.2 (a) $\left[t^3 - 4t\right]_0^2 = (8 - 8) - 0 = 0$. (b) The particle's **displacement** over $0 \le t \le 2$ is 0 — it returns to its start.

3.3 $\displaystyle\int_1^4 6t\,dt = \left[3t^2\right]_1^4 = 48 - 3 = 45$ L.

6. The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	0	2	3	6
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Part A

Find $\lim_{x \to 2} \dfrac{f(x)}{x}$, or state that the limit does not exist.

Part B

Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

Part C

Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

Part D

Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\,dt$.

i. Find $k'(x)$.

ii. Find $k''(3)$. Show the work that leads to your answer.

STOP

END OF EXAM

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

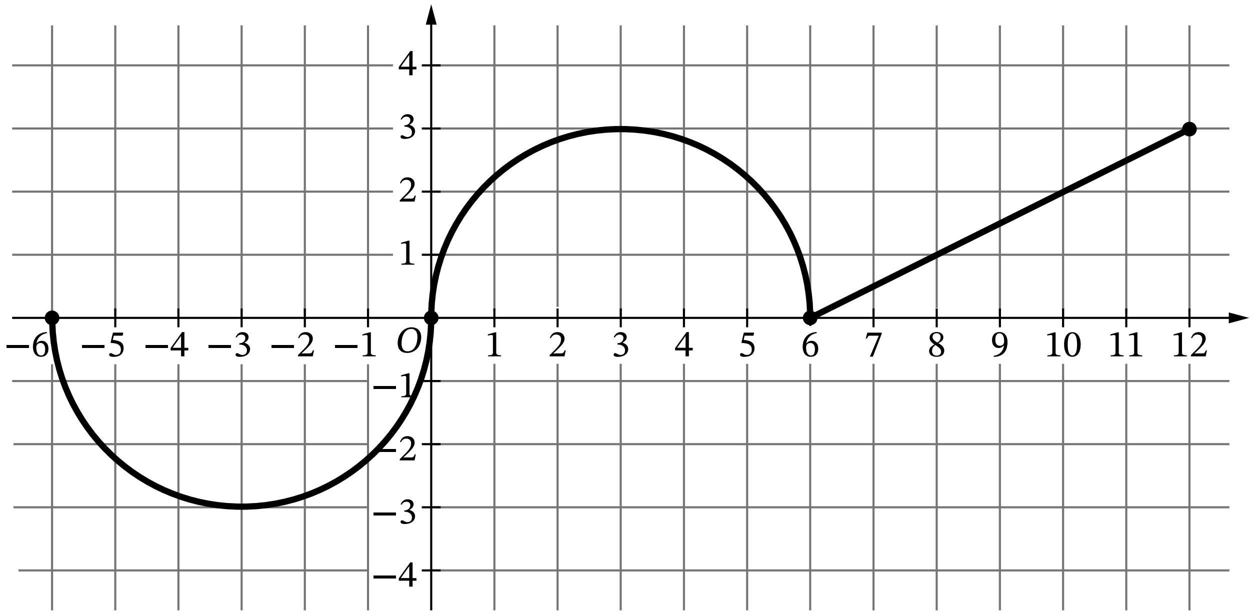
A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

4. The continuous function f is defined on the closed interval $-6 \le x \le 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_6^x f(t)\,dt$.

A. Find $g'(8)$. Give a reason for your answer.

B. Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.

C. Find $g(12)$ and $g(0)$. Label your answers.

D. Find the value of x at which g attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

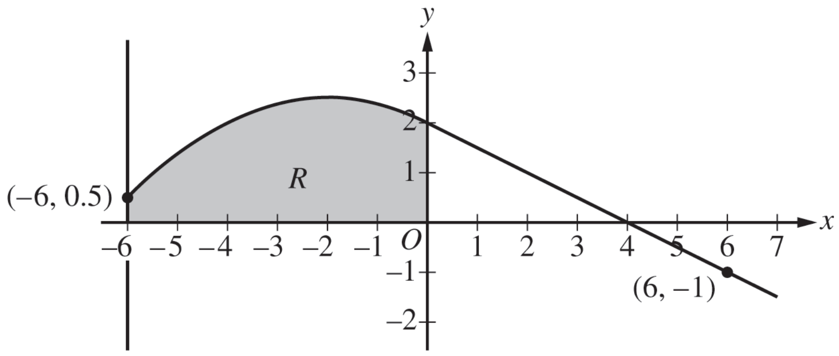
(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

(d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f

4. The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

5. The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

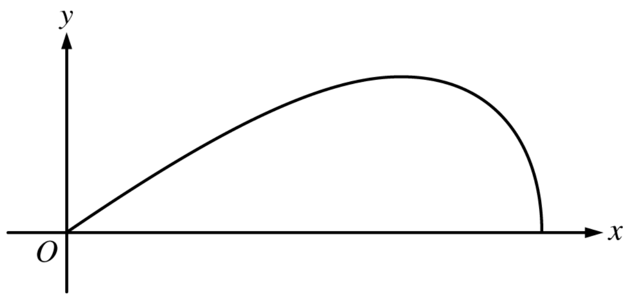
(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

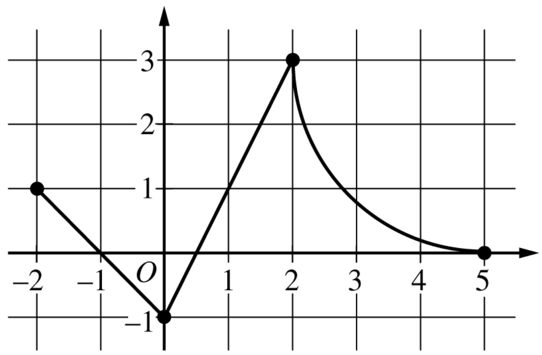
CALCULUS AB

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

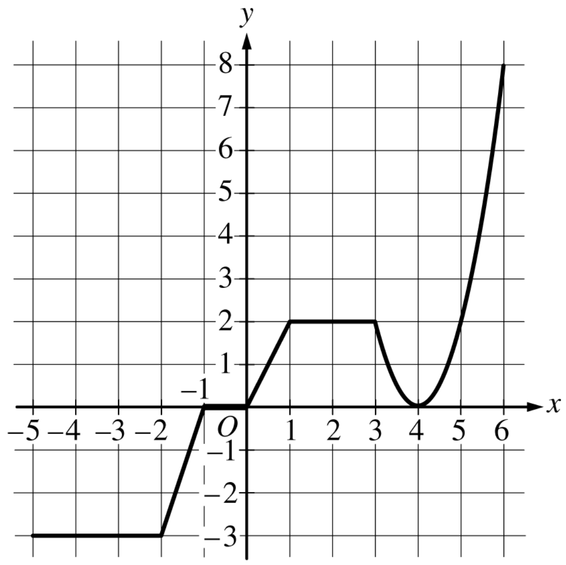
(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

CALCULUS AB
SECTION II, Part B
Time—1 hour
Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

   (a) If $f(1) = 3$, what is the value of $f(-5)$?

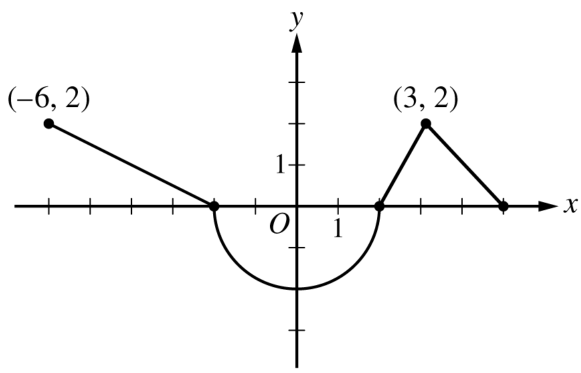
   (b) Evaluate $\int_1^6 g(x)\,dx$.

   (c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

   (d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

CALCULUS AB
SECTION II, Part B
Time—1 hour
Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

   (a) Find the values of $f(-6)$ and $f(5)$.

   (b) On what intervals is f increasing? Justify your answer.

   (c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

   (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	–6	3	2	8
2	2	–2	–3	0
3	8	7	6	2
6	4	5	3	–1

6. The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

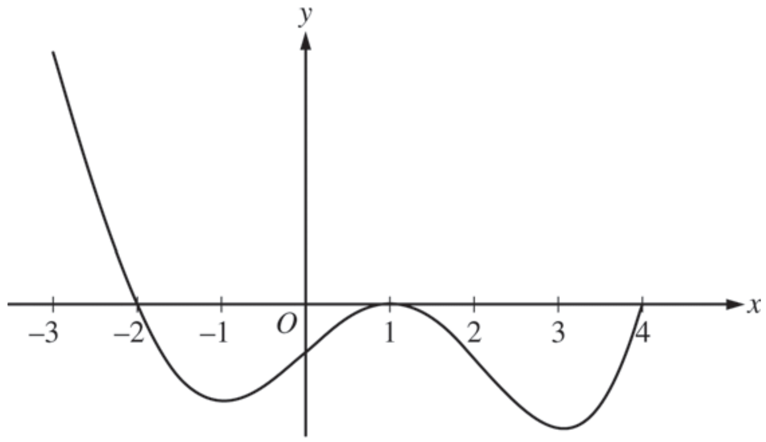
(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

(b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

STOP

END OF EXAM



Graph of f'

5. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

   (a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

   (b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

   (c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

   (d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

6.8 Antiderivatives and Indefinite Integrals

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS indefinite integral 不定积分 • constant of integration 积分常数 • antiderivative 原函数 • definite integral 定积分

Objective

Build the skills to answer exam questions on **antiderivatives and indefinite integrals** — reversing differentiation.

You must be able to:

- apply the **power rule for integration** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- integrate $\frac{1}{x}$, e^x, and basic trig functions
- always include the **constant of integration 积分常数** C

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The power rule for integration

Add one to the power and divide by the new power:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1).$$

So $\int x^3\,dx = \frac{x^4}{4} + C$ and $\int 5\,dx = 5x + C$.

■ The special case n = -1

The power rule fails for $n = -1$; instead $\int \frac{1}{x}\,dx = \ln|x| + C$.

■ Standard antiderivatives

$\int e^x\,dx = e^x + C$, $\int \cos x\,dx = \sin x + C$, $\int \sin x\,dx = -\cos x + C$, $\int \sec^2 x\,dx = \tan x + C$.

■ **Term by term (and never forget +C)**

Integrate a sum piece by piece: $\int (6x^2 - 4x + 1)\,dx = 2x^3 - 2x^2 + x + C$. The $+C$ is essential — an indefinite integral is a **family** of functions.

2 Practice

Now apply the methods above.

2.1 Find $\int x^5\,dx$. [1]

2.2 Find $\int (4x^3 - 2x)\,dx$. [2]

2.3 Find $\int \left(\frac{1}{x} + e^x\right)\,dx$. [2]

3 Exam-style questions

3.1 $\displaystyle\int \cos x \, dx =$ [1]

- **A** $-\sin x + C$
- **B** $\sin x + C$
- **C** $-\cos x + C$
- **D** $\tan x + C$

3.2 (a) Find the general antiderivative of $f(x) = 3x^2 + 4x$. [2]

(b) Find the particular antiderivative F with $F(0) = 5$. [2]

3.3 Find $\displaystyle\int \left(\sqrt{x} + \frac{2}{x}\right) dx$. Write $\sqrt{x}$ as $x^{1/2}$ first. [3]

Solutions

2.1 $\dfrac{x^6}{6} + C.$

2.2 $x^4 - x^2 + C.$

2.3 $\ln|x| + e^x + C.$

3.1 **B** — $\int \cos x \, dx = \sin x + C.$

3.2 (a) $F(x) = x^3 + 2x^2 + C$. (b) $F(0) = C = 5$, so $F(x) = x^3 + 2x^2 + 5$.

3.3 $\displaystyle\int (x^{1/2} + 2x^{-1}) \, dx = \frac{x^{3/2}}{3/2} + 2\ln|x| + C = \tfrac{2}{3}x^{3/2} + 2\ln|x| + C.$

6.9 Integrating Using Substitution

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS definite integral 定积分

Objective

Build the skills to answer exam questions on **integration by substitution** (u-substitution) — reversing the chain rule.

You must be able to:

- choose $u = g(x)$ so that its derivative $g'(x)$ also appears in the integrand
- convert dx to du and integrate in terms of u
- handle a **definite** integral by changing the limits or converting back to x

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Choosing u

Look for an "inside function" whose derivative is also present. For $\int 2x\,(x^2+1)^3\,dx$, let $u = x^2+1$, so $du = 2x\,dx$. The integral becomes

$$\int u^3\,du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{4} + C.$$

■ Adjusting for a constant

If the derivative is off by a constant, fix it. For $\int x\,(x^2+1)^3\,dx$, let $u = x^2+1$, $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$:

$$\int u^3 \cdot \tfrac{1}{2}\,du = \frac{u^4}{8} + C.$$

■ Definite integral — change the limits

For $\int_0^2 2x(x^2+1)^3\,dx$, with $u = x^2+1$: when $x = 0$, $u = 1$; when $x = 2$, $u = 5$. So

$$\int_1^5 u^3\,du = \left[\tfrac{u^4}{4}\right]_1^5 = \frac{625-1}{4} = 156.$$

A trig example

$\int \cos(3x)\,dx$: let $u = 3x$, $du = 3\,dx$, so $dx = \frac{1}{3}\,du$, giving $\frac{1}{3}\sin(3x) + C$.

2 Practice

Now apply the methods above.

2.1 For $\displaystyle\int 3x^2(x^3+5)^4\,dx$, state a good choice of u and find du. [2]

2.2 Find $\displaystyle\int 2x\,e^{x^2}\,dx$. [2]

2.3 Find $\displaystyle\int \sin(4x)\,dx$. [2]

3 Exam-style questions

3.1 To integrate $\displaystyle\int x\cos(x^2)\,dx$, the best substitution is [1]

- **A** $u = x$
- **B** $u = \cos x$
- **C** $u = x^2$
- **D** $u = \cos(x^2)$

3.2 Find $\displaystyle\int (2x+1)(x^2+x)^5\,dx$. [3]

3.3 Evaluate $\displaystyle\int_0^1 3x^2\sqrt{x^3+1}\,dx$ using a substitution, changing the limits. [4]

Solutions

2.1 $u = x^3 + 5$; $du = 3x^2\,dx$.

2.2 $u = x^2$, $du = 2x\,dx$; $\int e^u\,du = e^{x^2} + C$.

2.3 $u = 4x$, $dx = \frac{1}{4}du$; $-\frac{1}{4}\cos(4x) + C$.

3.1 **C** — $u = x^2$ gives $du = 2x\,dx$, matching the x in the integrand.

3.2 $u = x^2 + x$, $du = (2x+1)\,dx$; $\int u^5\,du = \frac{u^6}{6} = \dfrac{(x^2+x)^6}{6} + C$.

3.3 $u = x^3 + 1$, $du = 3x^2\,dx$; limits $x = 0 \to u = 1$, $x = 1 \to u = 2$; $\int_1^2 \sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_1^2$ $= \frac{2}{3}(2\sqrt{2} - 1) \approx 1.22$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	–6	3	2	8
2	2	–2	–3	0
3	8	7	6	2
6	4	5	3	–1

6. The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

   (a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

   (b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

   (c) Evaluate $\int_1^3 f''(2x)\,dx$.

STOP

END OF EXAM

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

6.10 Long Division and Completing the Square

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS complete the square 配方 • completing the square 配方法 • polynomial long division 多项式长除法

Objective

Build the skills to answer exam questions on **integrating with long division and completing the square** — rewriting a rational integrand into an integrable form.

You must be able to:

- use **polynomial long division 多项式长除法** when the numerator's degree is $\geq$ the denominator's
- **complete the square 配方** in a denominator to reach an **arctangent** form
- recognise $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Long division first

$\displaystyle\int \frac{x^2}{x+1}\,dx$ is top-heavy. Divide: $\displaystyle\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. Then

$$\int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{x^2}{2} - x + \ln|x+1| + C.$$

■ A simple 1/x-type

$\displaystyle\int \frac{1}{x-3}\,dx = \ln|x-3| + C$ (substitution $u = x - 3$). Watch for these after dividing.

■ Completing the square

$\displaystyle\int \frac{1}{x^2+4x+5}\,dx$: complete the square, $x^2 + 4x + 5 = (x+2)^2 + 1$. With $u = x + 2$,

$$\int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+2) + C.$$

■ **The arctangent template**

$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$. For $\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\arctan\frac{x}{3} + C$.

2 Practice

Now apply the methods above.

2.1 Use long division to rewrite $\frac{x+3}{x+1}$ as $1 + \frac{?}{x+1}$. [2]

2.2 Find $\int \frac{1}{x^2+16}\,dx$. [2]

2.3 Complete the square: write $x^2+6x+13$ in the form $(x+a)^2+b$. [2]

3 Exam-style questions

3.1 $\int \frac{1}{x^2+25}\,dx =$ [1]

- **A** $\arctan(x) + C$
- **B** $\frac{1}{5}\arctan\frac{x}{5} + C$
- **C** $\ln(x^2+25) + C$
- **D** $5\arctan(5x) + C$

3.2 Find $\int \frac{x+2}{x+1}\,dx$. [3]

3.3 Find $\int \frac{1}{x^2+2x+10}\,dx$ by completing the square. [4]

Solutions

2.1 $\dfrac{x+3}{x+1} = 1 + \dfrac{2}{x+1}$.

2.2 $\frac{1}{4}\arctan\frac{x}{4} + C$.

2.3 $(x+3)^2 + 4$.

3.1 **B** — with $a = 5$, $\int \frac{dx}{x^2+25} = \frac{1}{5}\arctan\frac{x}{5} + C$.

3.2 $\dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$; $\int \left(1 + \frac{1}{x+1}\right) dx = x + \ln|x+1| + C$.

3.3 $x^2+2x+10 = (x+1)^2+9$; with $u = x+1$, $\int \frac{du}{u^2+9} = \frac{1}{3}\arctan\frac{u}{3} = \frac{1}{3}\arctan\dfrac{x+1}{3} + C$.

6.14 Selecting Techniques for Antidifferentiation

Name: ____________ Class: ________ Date: __________ **Total: 10 marks**

KEY WORDS completing the square 配方法 • antiderivative 原函数

Objective

Build the skills to answer exam questions on **selecting antidifferentiation techniques** — choosing the right method quickly.

You must be able to:

- recognise when a **basic rule**, a **substitution**, **long division**, or **completing the square** applies
- match an integrand's shape to the fastest method
- carry the chosen method through to a correct antiderivative

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The decision guide

- **Basic rule** — a power, e^x, sin / cos, or $\frac{1}{x}$ on its own.
- **Substitution** — an inside function whose derivative also appears (e.g. $2x$ beside x^2+1).
- **Long division** — a rational function with numerator degree $\geq$ denominator degree.
- **Completing the square** — an irreducible quadratic x^2+bx+c in a denominator (arctangent).

Spotting a substitution

$\int x^2 e^{x^3}\,dx$: the x^2 is (up to a constant) the derivative of x^3 — **substitute** $u = x^3$. Answer $\frac{1}{3}e^{x^3} + C$.

■ **Spotting a division**

$\int \frac{x^3}{x^2+1}\,dx$: top-heavy, so **divide** first: $x - \frac{x}{x^2+1}$, then integrate (the second piece needs $u = x^2 + 1$).

■ **Spotting a basic rule**

$\int (2x + \cos x)\,dx$ needs no tricks — just the **basic rules**: $x^2 + \sin x + C$.

2 Practice

Now apply the methods above. For each, name the best method (basic / substitution / division / complete-the-square).

2.1 $\int \frac{1}{x^2+4}\,dx$ [1]

2.2 $\int 5x^4(x^5+2)^7\,dx$ [1]

2.3 $\int \frac{x^2+1}{x}\,dx$ [1]

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3 Exam-style questions

3.1 Which integral is best done by **substitution**? [1]

- **A** $\displaystyle\int (x^3 - 2)\,dx$
- **B** $\displaystyle\int \frac{1}{x^2+9}\,dx$
- **C** $\displaystyle\int \cos(x^2)\cdot 2x\,dx$
- **D** $\displaystyle\int \frac{x^2}{x+1}\,dx$

3.2 Find $\displaystyle\int \frac{x^2+3}{x}\,dx$ (rewrite the fraction first). [3]

3.3 Find $\displaystyle\int \frac{6x}{x^2+1}\,dx$. [3]

Solutions

2.1 Complete-the-square / arctangent form (basic arctangent rule).

2.2 Substitution, $u = x^5 + 2$.

2.3 Basic — split into $x + \frac{1}{x}$ first.

3.1 **C** — $2x$ is the derivative of x^2, so substitute $u = x^2$.

3.2 $\dfrac{x^2+3}{x} = x + \dfrac{3}{x}$; $\int \left(x + \frac{3}{x}\right) dx = \frac{x^2}{2} + 3\ln|x| + C$.

3.3 $u = x^2 + 1$, $du = 2x\,dx$, so $6x\,dx = 3\,du$; $\int \frac{3}{u}\,du = 3\ln|u| = 3\ln(x^2+1) + C$.