Week 12

# AP Calculus AB

Homework bundle for this week

本周作业合集

exercises.lol

## Contents 目录

AP Calculus AB Name: ____________ Score: ______

1. The MVT on $[a, b]$ requires **which** hypotheses?
   - ☐ $f$ continuous on $[a, b]$
   - ☐ $f$ differentiable on $(a, b)$
   - ☐ $f(a) = f(b)$
   - ☐ $f$ is a polynomial

   *Tick every correct option.*

2. A critical point is an interior $x$ where...
   - ☐ $f(x) = 0$
   - ☐ $f'(x) = 0$ or $f'(x)$ does not exist
   - ☐ $f(x)$ is largest
   - ☐ $f''(x) = 0$ only

3. A function is increasing on an interval exactly when...
   - ☐ $f(x) > 0$ there
   - ☐ $f'(x) > 0$ there
   - ☐ $f(x) < 0$ there
   - ☐ $f''(x) > 0$ there

4. If $f'$ changes from positive to negative at $c$, then $f$ has a...
   - ☐ local minimum
   - ☐ local maximum
   - ☐ inflection point
   - ☐ vertical asymptote

5. To find absolute extrema on $[a, b]$, evaluate $f$ at **which** points?
   - ☐ all critical points in $[a, b]$
   - ☐ the endpoint $a$
   - ☐ the endpoint $b$
   - ☐ every integer in $[a, b]$

   *Tick every correct option.*

6. A graph is concave up exactly where...



   - ☐ $f'(x) > 0$
   - ☐ $f''(x) > 0$
   - ☐ $f(x) > 0$
   - ☐ $f''(x) < 0$

7. If $f'(c) = 0$ and $f''(c) > 0$, then $c$ is a...
   - ☐ relative maximum
   - ☐ relative minimum
   - ☐ inflection point
   - ☐ endpoint

8. For $A(x) = 40x - 2x^2$, solve $A'(x) = 40 - 4x = 0$ for the critical $x$.

   ____________

9. For $f(x) = x^2$ on $[2, 6]$, find the $c$ guaranteed by the MVT.

   ____________

10. Find **all** critical points of $f(x) = x^3 - 3x$.
    - ☐ $x = 1$
    - ☐ $x = -1$
    - ☐ $x = 0$
    - ☐ $x = 3$

    *Tick every correct option.*

AP Calculus AB

1. **$f$ continuous on $[a, b]$; $f$ differentiable on $(a, b)$**

   Continuous on the closed interval and differentiable on the open one. ($f(a) = f(b)$ is the extra Rolle condition.)

2. **$f'(x) = 0$ or $f'(x)$ does not exist**

   Critical points: $f' = 0$ or $f'$ undefined.

3. **$f'(x) > 0$ there**

   Positive first derivative means increasing.

4. **local maximum**

   Up then down = the top of a hill = local max.

5. **all critical points in $[a, b]$; the endpoint $a$; the endpoint $b$**

   Critical points plus both endpoints — that is the full candidate list.

6. **$f''(x) > 0$**

   Concave up $\Leftrightarrow f'' > 0$ (slope increasing).

7. **relative minimum**

   Concave up at a flat spot = minimum.

8. **10**

   $40 - 4x = 0 \Rightarrow x = 10$.

9. **4**

   Average $= \frac{36-4}{4} = 8$; $2c = 8 \Rightarrow c = 4 \in (2, 6)$.

10. **$x = 1$; $x = -1$**

    $f'(x) = 3(x-1)(x+1) = 0$ at $x = \pm 1$.



AP CALCULUS AB • EXERCISE SHEET

# 5.1 The Mean Value Theorem

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

**KEY WORDS** mean value theorem 中值定理 • increasing 递增

## Objective

Build the skills to answer exam questions on **the Mean Value Theorem**.

**You must be able to:**

- state the **Mean Value Theorem 中值定理** and its hypotheses
- find the guaranteed point $c$

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

**The Mean Value Theorem**

If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, there is a $c$ in $(a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

— somewhere the tangent slope equals the average slope.

**Example**

$f(x) = x^2$ on $[0, 4]$: average slope $= \frac{16-0}{4} = 4$; $f'(c) = 2c = 4$ gives $c = 2$.

## 2 Practice

**2.1** State the MVT conclusion. [1]

**2.2** State the two hypotheses. [1]

**2.3** For $f(x) = x^2$ on $[0, 4]$, find the average rate of change. [2]

## 3 Exam-style questions

**3.1** The MVT guarantees a $c$ with $f'(c)$ equal to [1]

- **A** 0
- **B** the average rate of change on $[a, b]$
- **C** $f(a)$
- **D** $\infty$

**3.2** The MVT requires $f$ continuous on $[a, b]$ and [1]

- **A** $f = 0$
- **B** differentiable on $(a, b)$
- **C** linear
- **D** increasing

**3.3** $f(x) = x^2$ on $[0, 4]$.

(a) Find $f(0)$ and $f(4)$. [1]

(b) Find the average slope. [1]

(c) Solve $f'(c) =$ that for $c$. [1]



## Solutions

**2.1** there is a $c$ in $(a, b)$ where $f'(c)$ equals the average rate of change on $[a, b]$.

**2.2** $f$ continuous on $[a, b]$ and differentiable on $(a, b)$.

**2.3** $\dfrac{16 - 0}{4 - 0} = 4$.

**3.1** **B**.

**3.2** **B**.

**3.3** (a) $f(0) = 0$, $f(4) = 16$. (b) 4. (c) $2c = 4$, so $c = 2$.

**1.** An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function $C$ defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species $t$ weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25+t^2}$.

**(Note: Your calculator should be in radian mode.)**

**A.** Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

**B.** Find the time $t$ when the instantaneous rate of change of $C$ equals the average rate of change of $C$ over the time interval $0 \le t \le 4$. Show the setup for your calculations.

**C.** Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

**D.** At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function $A$, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time $t$, for $4 \le t \le 36$, does $A$ attain its maximum value? Justify your answer.





| $t$ (seconds) | 0 | 60 | 90 | 120 | 135 | 150 |
|---|---|---|---|---|---|---|
| $f(t)$ (gallons per second) | 0 | 0.1 | 0.15 | 0.1 | 0.05 | 0 |

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function $f$, where $f(t)$ is measured in gallons per second and $t$ is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

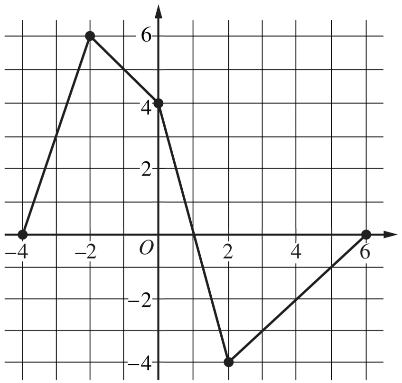
(b) Must there exist a value of $c$, for $60 < c < 120$, such that $f'(c) = 0$ ? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model $g$ defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**




Graph of $f$

. Let $f$ be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of $f$, consisting of four line segments, is shown above. Let $G$ be the function defined by $G(x) = \int_0^x f(t)\, dt$.

(a) On what open intervals is the graph of $G$ concave up? Give a reason for your answer.

(b) Let $P$ be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of $G$ on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value $c$, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

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**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**





| $t$ (hours) | 0 | 0.3 | 1.7 | 2.8 | 4 |
|---|---|---|---|---|---|
| $v_P(t)$ (meters per hour) | 0 | 55 | –29 | 55 | 48 |

2. The velocity of a particle, $P$, moving along the $x$-axis is given by the differentiable function $v_P$, where $v_P(t)$ is measured in meters per hour and $t$ is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle $P$ is at the origin at time $t = 0$.

(a) Justify why there must be at least one time $t$, for $0.3 \leq t \leq 2.8$, at which $v_P'(t)$, the acceleration of particle $P$, equals 0 meters per hour per hour.

(b) Use a trapezoidal sum with the three subintervals $[0, 0.3]$, $[0.3, 1.7]$, and $[1.7, 2.8]$ to approximate the value of $\int_0^{2.8} v_P(t)\, dt$.

(c) A second particle, $Q$, also moves along the $x$-axis so that its velocity for $0 \leq t \leq 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle $Q$ is at least 60 meters per hour. Find the distance traveled by particle $Q$ during the interval when the velocity of particle $Q$ is at least 60 meters per hour.

(d) At time $t = 0$, particle $Q$ is at position $x = -90$. Using the result from part (b) and the function $v_Q$ from part (c), approximate the distance between particles $P$ and $Q$ at time $t = 2.8$.

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**END OF PART A OF SECTION II**

| $t$ (years) | 2 | 3 | 5 | 7 | 10 |
|---|---|---|---|---|---|
| $H(t)$ (meters) | 1.5 | 2 | 6 | 11 | 15 |

4. The height of a tree at time $t$ is given by a twice-differentiable function $H$, where $H(t)$ is measured in meters and $t$ is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time $t$, for $2 < t < 10$, such that $H'(t) = 2$.

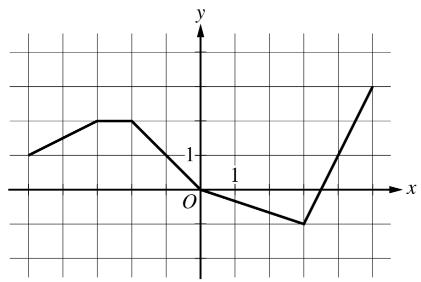
(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function $G$, given by $G(x) = \dfrac{100x}{1+x}$, where $x$ is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?





| $x$ | $g(x)$ | $g'(x)$ |
|---|---|---|
| –5 | 10 | –3 |
| –4 | 5 | –1 |
| –3 | 2 | 4 |
| –2 | 3 | 1 |
| –1 | 1 | –2 |
| 0 | 0 | –3 |




Graph of $h$

6. Let $f$ be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let $g$ be a differentiable function. The table above gives values of $g$ and its derivative $g'$ at selected values of $x$.

Let $h$ be the function whose graph, consisting of five line segments, is shown in the figure above.

(a) Find the slope of the line tangent to the graph of $f$ at $x = \pi$.

(b) Let $k$ be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

(c) Let $m$ be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

(d) Is there a number $c$ in the closed interval $[-5, -3]$ such that $g'(c) = -4$ ? Justify your answer.

**STOP**

**END OF EXAM**

| $x$ | $-2$ | $-2 < x < -1$ | $-1$ | $-1 < x < 1$ | $1$ | $1 < x < 3$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 12 | Positive | 8 | Positive | 2 | Positive | 7 |
| $f'(x)$ | $-5$ | Negative | 0 | Negative | 0 | Positive | $\frac{1}{2}$ |
| $g(x)$ | $-1$ | Negative | 0 | Positive | 3 | Positive | 1 |
| $g'(x)$ | 2 | Positive | $\frac{3}{2}$ | Positive | 0 | Negative | $-2$ |

5. The twice-differentiable functions $f$ and $g$ are defined for all real numbers $x$. Values of $f$, $f'$, $g$, and $g'$ for various values of $x$ are given in the table above.
   (a) Find the $x$-coordinate of each relative minimum of $f$ on the interval $[-2, 3]$. Justify your answers.
   (b) Explain why there must be a value $c$, for $-1 < c < 1$, such that $f''(c) = 0$.
   (c) The function $h$ is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.
   (d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.



AP CALCULUS AB • EXERCISE SHEET

# 5.2 Extreme Values and Critical Points

Name: ____________ Class: __________ Date: ____________ **Total: 9 marks**

**KEY WORDS** critical point 临界点 • extreme value theorem 极值定理

## Objective

Build the skills to answer exam questions on **the Extreme Value Theorem and critical points**.

**You must be able to:**

- state the **Extreme Value Theorem** 极值定理
- find the **critical points** 临界点 of a function

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

### EVT and critical points

**Extreme Value Theorem**: a function continuous on a closed interval $[a, b]$ attains a global maximum and minimum.

**Critical points** are where $f'(x) = 0$ or $f'$ does not exist — the candidates for local extrema.

### Example

$f(x) = x^2 - 4x$: $f'(x) = 2x - 4 = 0$ gives the critical point $x = 2$.

## 2 Practice

**2.1** State the Extreme Value Theorem. [1]

**2.2** State what a critical point is. [1]

**2.3** Find the critical point of $f(x) = x^2 - 4x$. [2]

## 3 Exam-style questions

**3.1** A critical point occurs where [1]

- **A** $f = 0$
- **B** $f' = 0$ or $f'$ does not exist
- **C** $f'' = 0$
- **D** $x = 0$

**3.2** The EVT guarantees extrema when $f$ is continuous on [1]

- **A** an open interval
- **B** a closed interval
- **C** all real numbers
- **D** no interval

**3.3** $f(x) = x^2 - 6x$.

(a) Find $f'(x)$. [1]

(b) Set $f' = 0$. [1]

(c) State the critical point. [1]



## Solutions

**2.1** a function continuous on a closed interval attains a global max and min.

**2.2** a point where $f' = 0$ or $f'$ does not exist.

**2.3** $f'(x) = 2x - 4 = 0$, so $x = 2$.

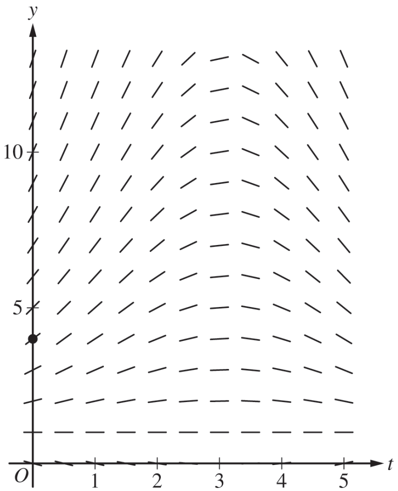
**3.1** **B**.

**3.2** **B**.

**3.3** (a) $2x - 6$. (b) $2x - 6 = 0$. (c) $x = 3$.

3. The depth of seawater at a location can be modeled by the function $H$ that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and $t$ is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



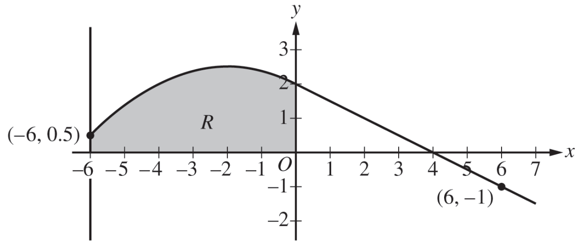

(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of $t$, for $0 < t < 5$, at which $H$ has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**








Graph of $f$

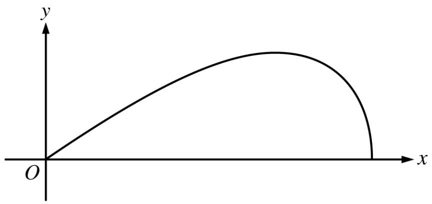
4. The graph of the differentiable function $f$, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let $R$ be the region in the second quadrant bounded by the graph of $f$, the vertical line $x = -6$, and the $x$- and $y$-axes. Region $R$ has area 12.

(a) The function $g$ is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function $g$ defined in part (a), find all values of $x$ in the interval $0 \le x \le 6$ at which the graph of $g$ has a critical point. Give a reason for your answer.

(c) The function $h$ is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**



. A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where $c$ is a positive constant. The figure above shows the region in the first quadrant bounded by the $x$-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some $c$. Each spinning toy is in the shape of the solid generated when such a region is revolved about the $x$-axis. Both $x$ and $y$ are measured in inches.

(a) Find the area of the region in the first quadrant bounded by the $x$-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of $c$ for this spinning toy?

(c) For another spinning toy, the volume is $2\pi$ cubic inches. What is the value of $c$ for this spinning toy?

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**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**





5. Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y \cos x}{4y - \sin x}$.

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether $f$ has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

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**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

AP CALCULUS AB • EXERCISE SHEET

# 5.3 Increasing and Decreasing Intervals

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

**KEY WORDS** increasing 递增 • decreasing 递减 • critical point 临界点

## Objective

Build the skills to answer exam questions on **intervals where a function is increasing or decreasing**.

**You must be able to:**

- use the sign of $f'$ to find where $f$ is **increasing 递增** or **decreasing 递减**

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

> ■ **Increasing and decreasing**
> - $f$ is **increasing** where $f' > 0$;
> - $f$ is **decreasing** where $f' < 0$.
>
> Test the sign of $f'$ on the intervals between critical points.
>
> ■ **Example**
>
> $f'(x) = 2x - 4$: $f' > 0$ for $x > 2$ (increasing) and $f' < 0$ for $x < 2$ (decreasing).

## 2 Practice

**2.1** State where $f$ is increasing. [1]

**2.2** For $f'(x) = 2x - 4$, find where $f$ is increasing. [2]



**2.3** For the same $f'$, state where $f$ is decreasing. [1]

## 3 Exam-style questions

**3.1** $f$ is increasing where [1]

- **A** $f' < 0$
- **B** $f' > 0$
- **C** $f'' > 0$
- **D** $f = 0$

**3.2** If $f'(x) = 2x - 4$, then $f$ is decreasing for [1]

- **A** $x > 2$
- **B** $x < 2$
- **C** all $x$
- **D** $x = 2$

**3.3** $f'(x) = 3x^2 - 3$.

(a) Solve $f' = 0$. [1]

(b) Test a point in $(-1, 1)$. [1]

(c) State where $f$ is decreasing. [1]

## Solutions

**2.1** where $f' > 0$.

**2.2** $2x - 4 > 0 \Rightarrow x > 2$.

**2.3** $x < 2$.

**3.1** **B**.

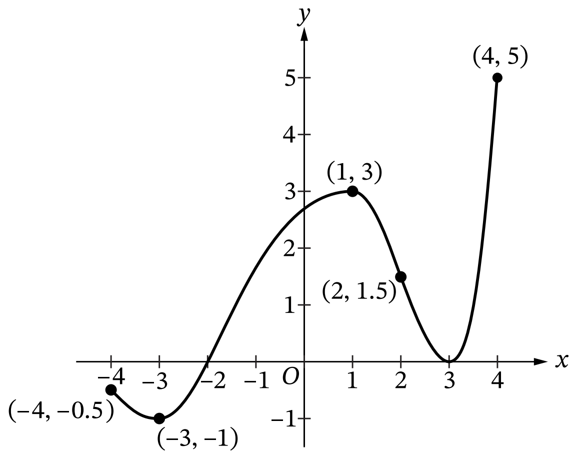
**3.2** **B**.

**3.3** (a) $x = \pm 1$. (b) $f'(0) = -3 < 0$. (c) $(-1, 1)$.





**4.** Let $f$ be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of $f'$, the derivative of $f$, is shown.




Graph of $f'$

**Part A**

For $x > 0$, the function $g$ is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

**Part B**

Find all values of $x$ on the open interval $0 < x < 3$ at which the graph of $f$ has a point of inflection. Give a reason for your answer.

**Part C**

For $-4 \le x \le 4$, on what open intervals, if any, is the graph of $f$ both increasing and concave down? Give a reason for your answer.

**Part D**

For $-4 \le x \le 4$, find the value of $x$ at which $f$ has an absolute minimum and the value of $x$ at which $f$ has an absolute maximum. Give reasons for your answers.

| $x$ | 0 | 2 | 4 | 7 |
|---|---|---|---|---|
| $f(x)$ | 10 | 7 | 4 | 5 |
| $f'(x)$ | $\frac{3}{2}$ | $-8$ | 3 | 6 |
| $g(x)$ | 1 | 2 | $-3$ | 0 |
| $g'(x)$ | 5 | 4 | 2 | 8 |

5. The functions $f$ and $g$ are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of $x$.

   (a) Let $h$ be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

   (b) Let $k$ be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of $k$ concave up or concave down at the point where $x = 4$ ? Give a reason for your answer.

   (c) Let $m$ be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

   (d) Is the function $m$ defined in part (c) increasing, decreasing, or neither at $x = 2$ ? Justify your answer.

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**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**





1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where $t$ is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time $t$, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\,dx$, where $a$ is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

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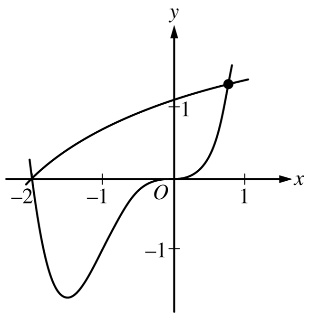
**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**




2. Let $f$ and $g$ be the functions defined by $f(x) = \ln(x + 3)$ and $g(x) = x^4 + 2x^3$. The graphs of $f$ and $g$, shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.

   (a) Find the area of the region enclosed by the graphs of $f$ and $g$.

   (b) For $-2 \le x \le B$, let $h(x)$ be the vertical distance between the graphs of $f$ and $g$. Is $h$ increasing or decreasing at $x = -0.5$ ? Give a reason for your answer.

   (c) The region enclosed by the graphs of $f$ and $g$ is the base of a solid. Cross sections of the solid taken perpendicular to the $x$-axis are squares. Find the volume of the solid.

   (d) A vertical line in the $xy$-plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of 7 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**





**END OF PART A**

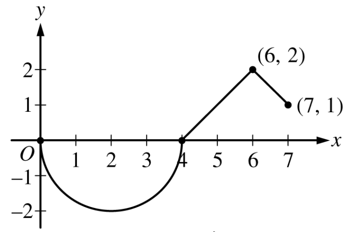
**CALCULUS AB**

**SECTION II, Part B**

**Time—1 hour**

**4 Questions**

**NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.**




Graph of $f'$

3. Let $f$ be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of $f'$, the derivative of $f$, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

(b) Find the $x$-coordinates of all points of inflection of the graph of $f$ for $0 < x < 7$. Justify your answer.

(c) Let $g$ be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is $g$ decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

(d) For the function $g$ defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

---

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**





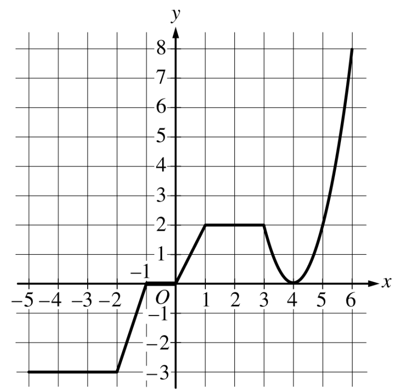
**CALCULUS AB**

**SECTION II, Part B**

**Time—1 hour**

**Number of questions—4**

**NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.**




Graph of $g$

3. The graph of the continuous function $g$, the derivative of the function $f$, is shown above. The function $g$ is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x - 4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$ ?

(b) Evaluate $\int_1^6 g(x)\, dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of $f$ both increasing and concave up? Give a reason for your answer.

(d) Find the $x$-coordinate of each point of inflection of the graph of $f$. Give a reason for your answer.

---

2. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and $t$ is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and $t$ is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

(b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

(c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$ ? Give a reason for your answer.

(d) How many pounds of bananas are on the display table at time $t = 8$ ?

---

**END OF PART A OF SECTION II**





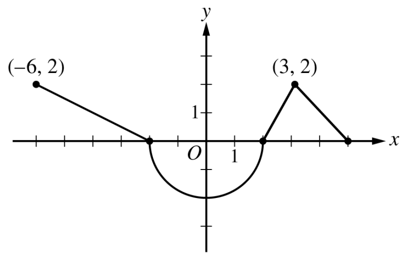
**CALCULUS AB**

**SECTION II, Part B**

**Time—1 hour**

**Number of questions—4**

**NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.**




Graph of $f'$

3. The function $f$ is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of $f'$, the derivative of $f$, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is $f$ increasing? Justify your answer.

(c) Find the absolute minimum value of $f$ on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

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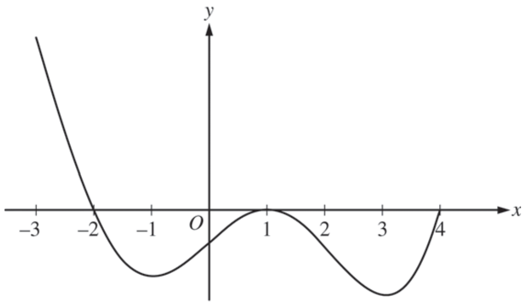



Graph of $f'$

5. The figure above shows the graph of $f'$, the derivative of a twice-differentiable function $f$, on the interval $[-3, 4]$. The graph of $f'$ has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the $x$-axis and the graph of $f'$ on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

   (a) Find all $x$-coordinates at which $f$ has a relative maximum. Give a reason for your answer.

   (b) On what open intervals contained in $-3 < x < 4$ is the graph of $f$ both concave down and decreasing? Give a reason for your answer.

   (c) Find the $x$-coordinates of all points of inflection for the graph of $f$. Give a reason for your answer.

   (d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.



**AP CALCULUS AB • EXERCISE SHEET**

# 5.4 The First Derivative Test

Name: ____________ Class: __________ Date: ____________ **Total: 9 marks**

**KEY WORDS** local minimum 极小值 • local maximum 极大值 • local extremum 局部极值 • critical point 临界点

## Objective

Build the skills to answer exam questions on **the First Derivative Test**.

**You must be able to:**

- use the sign change of $f'$ to classify a **local extremum** 局部极值

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

**■ First Derivative Test**

At a critical point, look at how $f'$ changes:

- $+$ to $-$ $\to$ local **maximum**;
- $-$ to $+$ $\to$ local **minimum**;
- no sign change $\to$ neither.

**■ Example**

$f'(x) = 2x - 4$ at $x = 2$: $f'$ goes $-$ (for $x < 2$) to $+$ (for $x > 2$), so $x = 2$ is a local minimum.

## 2 Practice

**2.1** State the sign change for a local maximum. [1]

**2.2** State the sign change for a local minimum. [1]

**2.3** For $f'(x) = 2x - 4$, classify $x = 2$. [2]

## 3 Exam-style questions

**3.1** A local maximum occurs where $f'$ changes [1]

- **A** $-$ to $+$
- **B** $+$ to $-$
- **C** stays $+$
- **D** stays $-$

**3.2** A local minimum occurs where $f'$ changes [1]

- **A** $+$ to $-$
- **B** $-$ to $+$
- **C** stays 0
- **D** never

**3.3** $f'(x) = 2x - 4$, critical at $x = 2$.

(a) State the sign of $f'$ for $x < 2$. [1]

(b) State the sign for $x > 2$. [1]

(c) Classify $x = 2$. [1]



## Solutions

**2.1** $f'$ changes from $+$ to $-$.

**2.2** $f'$ changes from $-$ to $+$.

**2.3** $f'$ goes $-$ to $+$, so $x = 2$ is a local minimum.

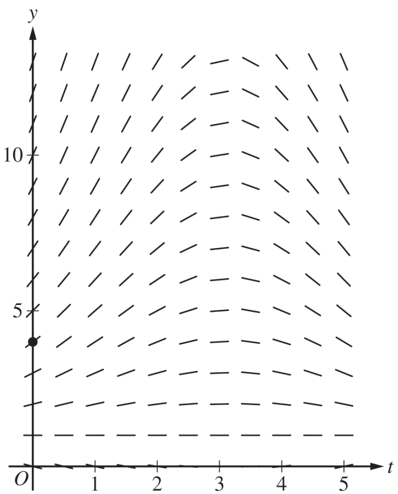
**3.1** **B**.

**3.2** **B**.

**3.3** (a) negative. (b) positive. (c) local minimum.

3. The depth of seawater at a location can be modeled by the function $H$ that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and $t$ is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

   (a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



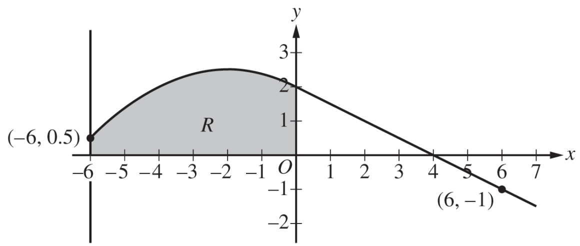

   (b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of $t$, for $0 < t < 5$, at which $H$ has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

   (c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**








Graph of $f$

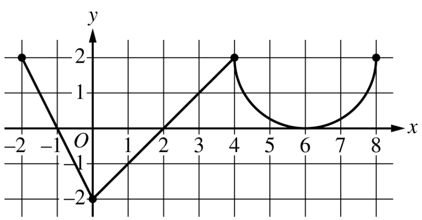
4. The graph of the differentiable function $f$, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let $R$ be the region in the second quadrant bounded by the graph of $f$, the vertical line $x = -6$, and the $x$- and $y$-axes. Region $R$ has area 12.

   (a) The function $g$ is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

   (b) For the function $g$ defined in part (a), find all values of $x$ in the interval $0 \le x \le 6$ at which the graph of $g$ has a critical point. Give a reason for your answer.

   (c) The function $h$ is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**




Graph of $f'$

4. The function $f$ is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of $f'$, the derivative of $f$, consists of two line segments and a semicircle, as shown in the figure.

(a) Does $f$ have a relative minimum, a relative maximum, or neither at $x = 6$ ? Give a reason for your answer.

(b) On what open intervals, if any, is the graph of $f$ concave down? Give a reason for your answer.

(c) Find the value of $\lim\limits_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of $f$ on the closed interval $[-2, 8]$. Justify your answer.

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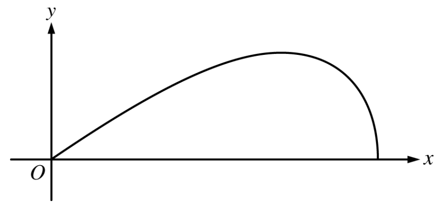
**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**








A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where $c$ is a positive constant. The figure above shows the region in the first quadrant bounded by the $x$-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some $c$. Each spinning toy is in the shape of the solid generated when such a region is revolved about the $x$-axis. Both $x$ and $y$ are measured in inches.

(a) Find the area of the region in the first quadrant bounded by the $x$-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of $c$ for this spinning toy?

(c) For another spinning toy, the volume is $2\pi$ cubic inches. What is the value of $c$ for this spinning toy?

---

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

5. Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y \cos x}{4y - \sin x}$.

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether $f$ has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

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**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**





**CALCULUS AB**

**SECTION II, Part A**

**Time—30 minutes**

**Number of questions—2**

**A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.**

1. People enter a line for an escalator at a rate modeled by the function $r$ given by

$$r(t) = \begin{cases} 44\left(\dfrac{t}{100}\right)^3\left(1 - \dfrac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and $t$ is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$ ?

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$ ?

(c) For $t > 300$, what is the first time $t$ that there are no people in line for the escalator?

(d) For $0 \le t \le 300$, at what time $t$ is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

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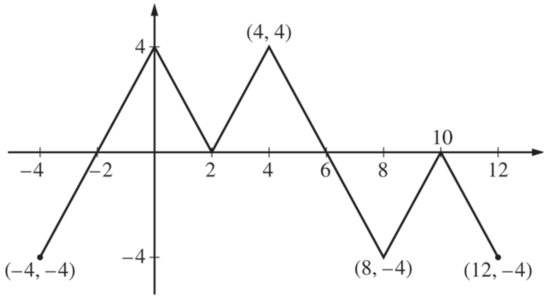
**CALCULUS AB**

**SECTION II, Part B**

**Time—60 minutes**

**Number of problems—4**

**No calculator is allowed for these problems.**




Graph of $f$

3. The figure above shows the graph of the piecewise-linear function $f$. For $-4 \le x \le 12$, the function $g$ is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 10$ ? Justify your answer.

(b) Does the graph of $g$ have a point of inflection at $x = 4$ ? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of $g$ on the interval $-4 \le x \le 12$. Justify your answers.

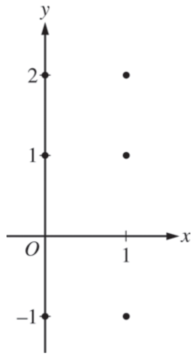
(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.





4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

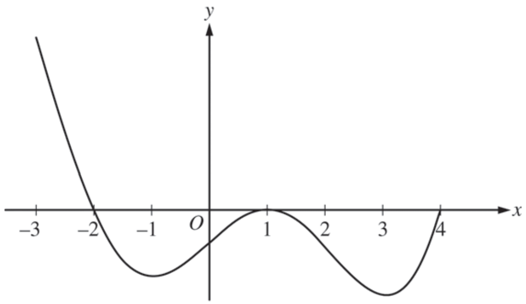
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.




(b) Find $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does $f$ have a relative minimum, a relative maximum, or neither at $x = 2$ ? Justify your answer.

(d) Find the values of the constants $m$ and $b$ for which $y = mx + b$ is a solution to the differential equation.




Graph of $f'$

5. The figure above shows the graph of $f'$, the derivative of a twice-differentiable function $f$, on the interval $[-3, 4]$. The graph of $f'$ has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the $x$-axis and the graph of $f'$ on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all $x$-coordinates at which $f$ has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of $f$ both concave down and decreasing? Give a reason for your answer.

(c) Find the $x$-coordinates of all points of inflection for the graph of $f$. Give a reason for your answer.

(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.





| $x$ | $-2$ | $-2 < x < -1$ | $-1$ | $-1 < x < 1$ | $1$ | $1 < x < 3$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 12 | Positive | 8 | Positive | 2 | Positive | 7 |
| $f'(x)$ | $-5$ | Negative | 0 | Negative | 0 | Positive | $\frac{1}{2}$ |
| $g(x)$ | $-1$ | Negative | 0 | Positive | 3 | Positive | 1 |
| $g'(x)$ | 2 | Positive | $\frac{3}{2}$ | Positive | 0 | Negative | $-2$ |

5. The twice-differentiable functions $f$ and $g$ are defined for all real numbers $x$. Values of $f$, $f'$, $g$, and $g'$ for various values of $x$ are given in the table above.

(a) Find the $x$-coordinate of each relative minimum of $f$ on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value $c$, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function $h$ is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

# 5.5 The Candidates Test for Absolute Extrema

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

**KEY WORDS** absolute (global) extrema 绝对极值 • critical point 临界点

## Objective

Build the skills to answer exam questions on **the Candidates Test for absolute extrema**.

**You must be able to:**

- find the **absolute (global) extrema 绝对极值** on a closed interval

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

### The Candidates Test

On a closed interval $[a, b]$, evaluate $f$ at **all critical points and both endpoints**. The largest value is the absolute maximum; the smallest is the absolute minimum.

### Example

$f(x) = x^2$ on $[-1, 3]$: critical point $x = 0$; candidates $f(-1) = 1$, $f(0) = 0$, $f(3) = 9$; absolute max 9, absolute min 0.

## 2 Practice

**2.1** State what to evaluate for absolute extrema on $[a, b]$. [1]

**2.2** For $f(x) = x^2$ on $[-1, 3]$, list the candidate inputs. [2]



**2.3** State how to pick the absolute maximum. [1]

## 3 Exam-style questions

**3.1** To find absolute extrema on $[a, b]$, evaluate $f$ at [1]

- **A** only the endpoints
- **B** the critical points and endpoints
- **C** only the critical points
- **D** $x = 0$

**3.2** The absolute maximum is the [1]

- **A** smallest candidate value
- **B** largest candidate value
- **C** midpoint
- **D** average

**3.3** $f(x) = x^2$ on $[-1, 3]$.

(a) State the critical point. [1]

(b) Evaluate at the candidates $-1, 0, 3$. [1]

(c) State the absolute maximum. [1]

## Solutions

**2.1** $f$ at all critical points and the two endpoints.

**2.2** $x = -1$, $x = 0$, $x = 3$.

**2.3** the largest of the candidate values.

**3.1** **B**.

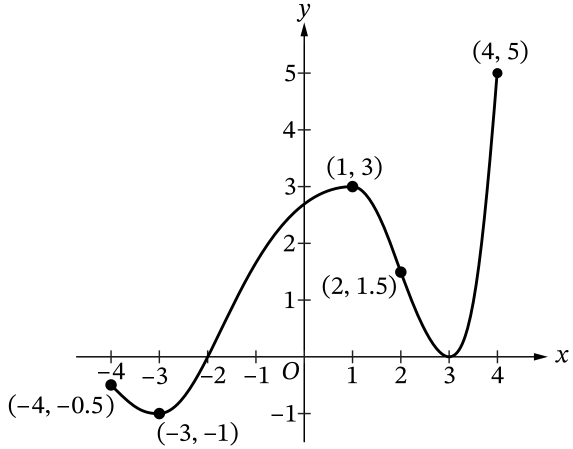
**3.2** **B**.

**3.3** (a) $x = 0$. (b) $f(-1) = 1$, $f(0) = 0$, $f(3) = 9$. (c) 9.





**4.** Let $f$ be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of $f'$, the derivative of $f$, is shown.




Graph of $f'$

**Part A**

For $x > 0$, the function $g$ is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

**Part B**

Find all values of $x$ on the open interval $0 < x < 3$ at which the graph of $f$ has a point of inflection. Give a reason for your answer.

**Part C**

For $-4 \le x \le 4$, on what open intervals, if any, is the graph of $f$ both increasing and concave down? Give a reason for your answer.

**Part D**

For $-4 \le x \le 4$, find the value of $x$ at which $f$ has an absolute minimum and the value of $x$ at which $f$ has an absolute maximum. Give reasons for your answers.

**1.** An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function $C$ defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species $t$ weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

**(Note: Your calculator should be in radian mode.)**

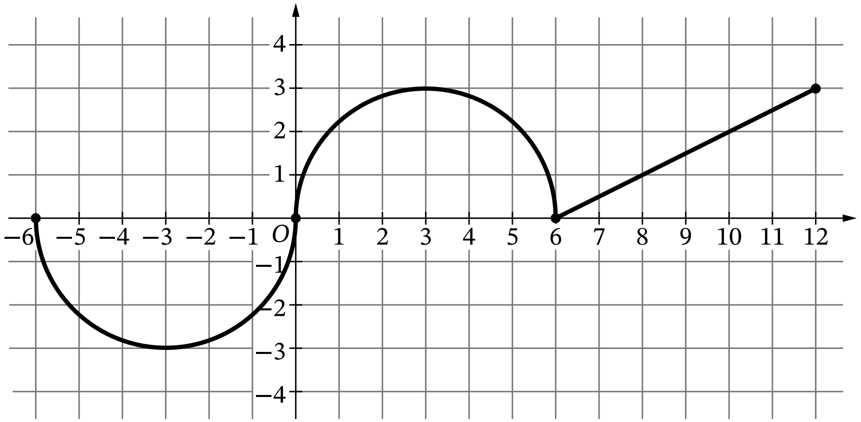
**A.** Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

**B.** Find the time $t$ when the instantaneous rate of change of $C$ equals the average rate of change of $C$ over the time interval $0 \le t \le 4$. Show the setup for your calculations.

**C.** Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

**D.** At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function $A$, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time $t$, for $4 \le t \le 36$, does $A$ attain its maximum value? Justify your answer.

**4.** The continuous function $f$ is defined on the closed interval $-6 \le x \le 12$. The graph of $f$, consisting of two semicircles and one line segment, is shown in the figure.



Graph of $f$

Let $g$ be the function defined by $g(x) = \int_6^x f(t)\,dt$.

**A.** Find $g'(8)$. Give a reason for your answer.

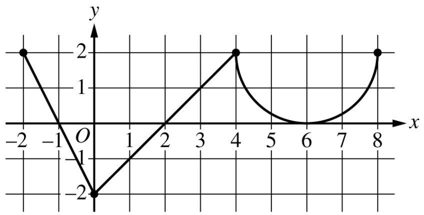
**B.** Find all values of $x$ in the open interval $-6 < x < 12$ at which the graph of $g$ has a point of inflection. Give a reason for your answer.

**C.** Find $g(12)$ and $g(0)$. Label your answers.

**D.** Find the value of $x$ at which $g$ attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.







Graph of $f'$

4. The function $f$ is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of $f'$, the derivative of $f$, consists of two line segments and a semicircle, as shown in the figure.

(a) Does $f$ have a relative minimum, a relative maximum, or neither at $x = 6$ ? Give a reason for your answer.

(b) On what open intervals, if any, is the graph of $f$ concave down? Give a reason for your answer.

(c) Find the value of $\lim\limits_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of $f$ on the closed interval $[-2, 8]$. Justify your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where $t$ is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

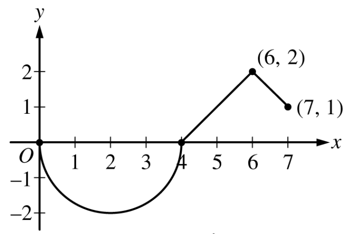
   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time $t$, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where $a$ is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**








Graph of $f'$

3. Let $f$ be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of $f'$, the derivative of $f$, consists of a semicircle and two line segments, as shown in the figure above.

   (a) Find $f(0)$ and $f(5)$.

   (b) Find the $x$-coordinates of all points of inflection of the graph of $f$ for $0 < x < 7$. Justify your answer.

   (c) Let $g$ be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is $g$ decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.

   (d) For the function $g$ defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

**CALCULUS AB**

**SECTION II, Part A**

**Time—30 minutes**

**Number of questions—2**

**A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.**

1. Fish enter a lake at a rate modeled by the function $E$ given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function $L$ given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and $t$ is measured in hours since midnight $(t = 0)$.

(a) How many fish enter the lake over the 5-hour period from midnight $(t = 0)$ to 5 A.M. $(t = 5)$ ? Give your answer to the nearest whole number.

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight $(t = 0)$ to 5 A.M. $(t = 5)$ ?

(c) At what time $t$, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. $(t = 5)$ ? Explain your reasoning.





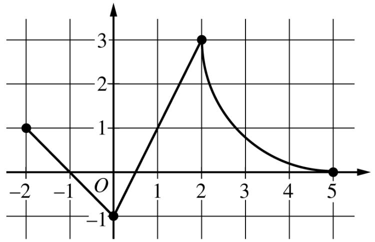
**CALCULUS AB**

**SECTION II, Part B**

**Time—1 hour**

**Number of questions—4**

**NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.**




Graph of $f$

3. The continuous function $f$ is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of $f$, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of $f$.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function $g$ is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of $g$ on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

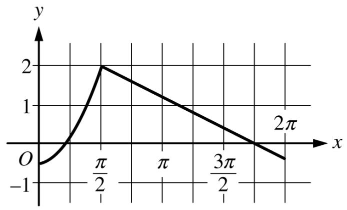
5. Let $f$ be the function defined by $f(x) = e^x \cos x$.

   (a) Find the average rate of change of $f$ on the interval $0 \le x \le \pi$.

   (b) What is the slope of the line tangent to the graph of $f$ at $x = \dfrac{3\pi}{2}$ ?

   (c) Find the absolute minimum value of $f$ on the interval $0 \le x \le 2\pi$. Justify your answer.

   (d) Let $g$ be a differentiable function such that $g\left(\dfrac{\pi}{2}\right) = 0$. The graph of $g'$, the derivative of $g$, is shown below. Find the value of $\lim\limits_{x \to \pi/2} \dfrac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.




Graph of $g'$





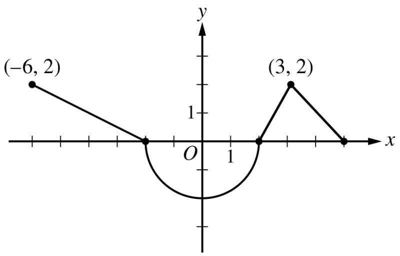
**CALCULUS AB**

**SECTION II, Part B**

**Time—1 hour**

**Number of questions—4**

**NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.**




Graph of $f'$

3. The function $f$ is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of $f'$, the derivative of $f$, consists of a semicircle and three line segments, as shown in the figure above.

   (a) Find the values of $f(-6)$ and $f(5)$.

   (b) On what intervals is $f$ increasing? Justify your answer.

   (c) Find the absolute minimum value of $f$ on the closed interval $[-6, 5]$. Justify your answer.

   (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

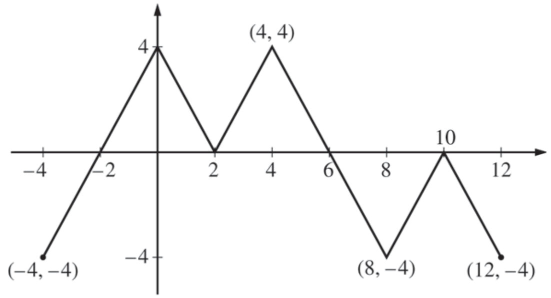
**CALCULUS AB**

**SECTION II, Part B**

**Time—60 minutes**

**Number of problems—4**

**No calculator is allowed for these problems.**




Graph of $f$

3. The figure above shows the graph of the piecewise-linear function $f$. For $-4 \le x \le 12$, the function $g$ is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 10$ ? Justify your answer.

(b) Does the graph of $g$ have a point of inflection at $x = 4$ ? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of $g$ on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.





**CALCULUS AB**

**SECTION II, Part A**

**Time—30 minutes**

**Number of problems—2**

**A graphing calculator is required for these problems.**

1. The rate at which rainwater flows into a drainpipe is modeled by the function $R$, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, $t$ is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$ ?

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

(c) At what time $t$, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time $w$ when the pipe will begin to overflow.

AP CALCULUS AB • EXERCISE SHEET

# 5.6 Concavity and Points of Inflection

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

**KEY WORDS** concave down 下凹 • concave up 上凹 • inflection point 拐点 • concavity 凹凸性

## Objective

Build the skills to answer exam questions on **determining concavity**.

**You must be able to:**

- use the sign of $f''$ to find **concavity** 凹凸性
- identify an **inflection point** 拐点

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Concavity**

- $f$ is **concave up** where $f'' > 0$;
- $f$ is **concave down** where $f'' < 0$.

An **inflection point** is where concavity changes (and $f''$ changes sign).

■ **Example**

$f(x) = x^3$: $f''(x) = 6x$, which is $> 0$ for $x > 0$; the inflection point is at $x = 0$.

## 2 Practice

**2.1** State where $f$ is concave up. [1]

**2.2** For $f(x) = x^3$, find $f''$ and where it is concave up. [2]



**2.3** State what an inflection point is. [1]

## 3 Exam-style questions

**3.1** $f$ is concave up where [1]

- **A** $f' > 0$
- **B** $f'' > 0$
- **C** $f'' < 0$
- **D** $f = 0$

**3.2** An inflection point is where [1]

- **A** $f' = 0$
- **B** concavity changes
- **C** $f = 0$
- **D** $f''$ is constant

**3.3** $f(x) = x^3$.

(a) Find $f''(x)$. [1]

(b) State where $f'' > 0$. [1]

(c) State the inflection point. [1]

## Solutions

**2.1** where $f'' > 0$.

**2.2** $f''(x) = 6x$; concave up for $x > 0$.

**2.3** a point where concavity changes (from up to down or vice versa).

**3.1** **B**.

**3.2** **B**.

**3.3** (a) $6x$. (b) $x > 0$. (c) $x = 0$.

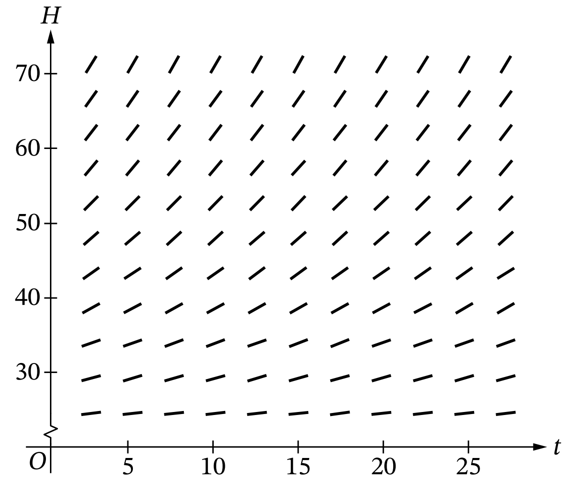




**3.** A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time $t$ minutes can be modeled by the function $H$ that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

**Part A**

Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.




**Part B**

Find the slope of the line tangent to the graph of $H$ at time $t = 0$. Show the work that leads to your answer.

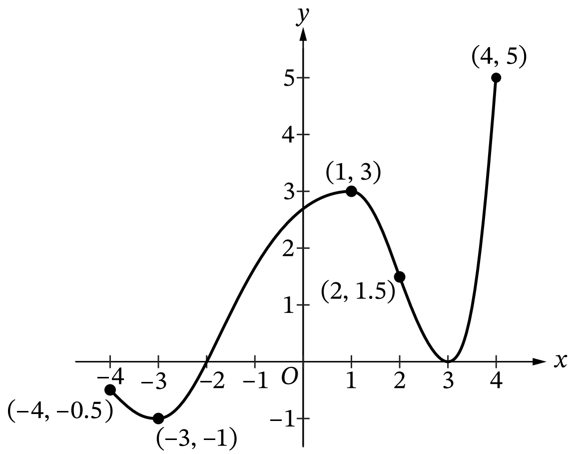
**Part C**

It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of $H$ at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

**Part D**

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

**4.** Let $f$ be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of $f'$, the derivative of $f$, is shown.




Graph of $f'$

**Part A**

For $x > 0$, the function $g$ is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

**Part B**

Find all values of $x$ on the open interval $0 < x < 3$ at which the graph of $f$ has a point of inflection. Give a reason for your answer.

**Part C**

For $-4 \le x \le 4$, on what open intervals, if any, is the graph of $f$ both increasing and concave down? Give a reason for your answer.

**Part D**

For $-4 \le x \le 4$, find the value of $x$ at which $f$ has an absolute minimum and the value of $x$ at which $f$ has an absolute maximum. Give reasons for your answers.





| $t$ (minutes) | 0 | 3 | 7 | 12 |
|---|---|---|---|---|
| $C(t)$ (degrees Celsius) | 100 | 85 | 69 | 55 |

1. The temperature of coffee in a cup at time $t$ minutes is modeled by a decreasing differentiable function $C$, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

(a) Approximate $C'(5)$ using the average rate of change of $C$ over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

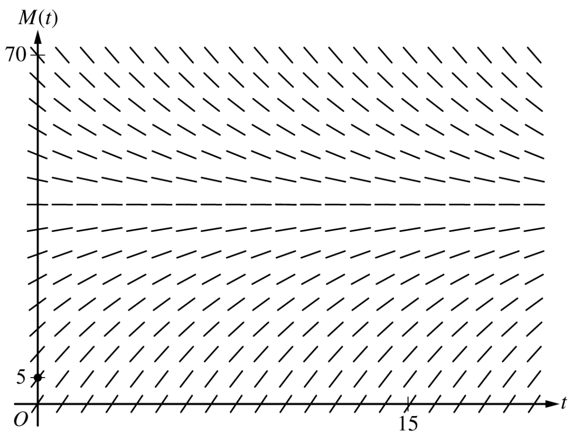
(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

(d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function $M$ models the temperature of the milk at time $t$, where $M(t)$ is measured in degrees Celsius (°C) and $t$ is the number of minutes since the bottle was placed in the pan. $M$ satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of $t$.

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.




(b) Use the line tangent to the graph of $M$ at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

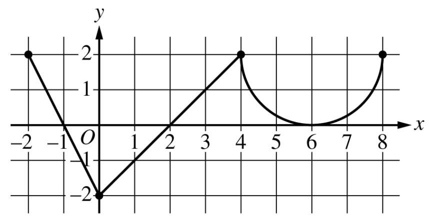
(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of $M$. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**








Graph of $f'$

4. The function $f$ is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of $f'$, the derivative of $f$, consists of two line segments and a semicircle, as shown in the figure.

(a) Does $f$ have a relative minimum, a relative maximum, or neither at $x = 6$ ? Give a reason for your answer.

(b) On what open intervals, if any, is the graph of $f$ concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of $f$ on the closed interval $[-2, 8]$. Justify your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

| $x$ | 0 | 2 | 4 | 7 |
|---|---|---|---|---|
| $f(x)$ | 10 | 7 | 4 | 5 |
| $f'(x)$ | $\frac{3}{2}$ | $-8$ | 3 | 6 |
| $g(x)$ | 1 | 2 | $-3$ | 0 |
| $g'(x)$ | 5 | 4 | 2 | 8 |

5. The functions $f$ and $g$ are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of $x$.

(a) Let $h$ be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let $k$ be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of $k$ concave up or concave down at the point where $x = 4$ ? Give a reason for your answer.

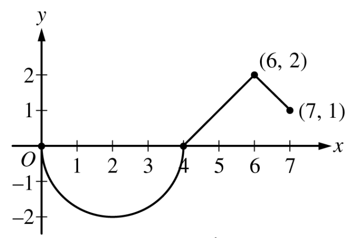
(c) Let $m$ be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

(d) Is the function $m$ defined in part (c) increasing, decreasing, or neither at $x = 2$ ? Justify your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**








Graph of $f'$

3. Let $f$ be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of $f'$, the derivative of $f$, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

(b) Find the $x$-coordinates of all points of inflection of the graph of $f$ for $0 < x < 7$. Justify your answer.

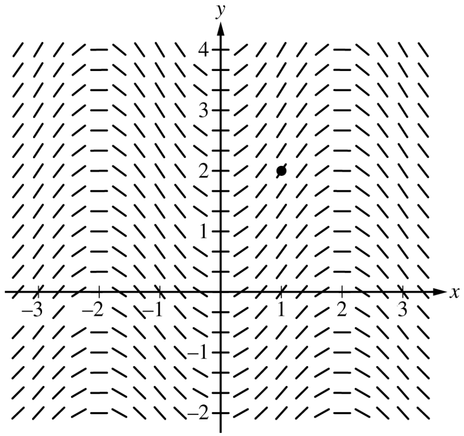
(c) Let $g$ be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is $g$ decreasing for $0 \le x \le 7$ ? Show the analysis that leads to your answer.

(d) For the function $g$ defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function $f$ is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.




(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

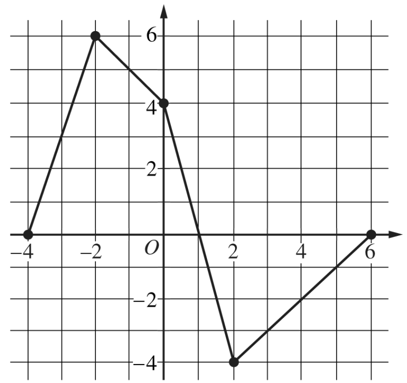
(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**








Graph of $f$

Let $f$ be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of $f$, consisting of four line segments, is shown above. Let $G$ be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of $G$ concave up? Give a reason for your answer.

(b) Let $P$ be the function defined by $P(x) = G(x)\cdot f(x)$. Find $P'(3)$.

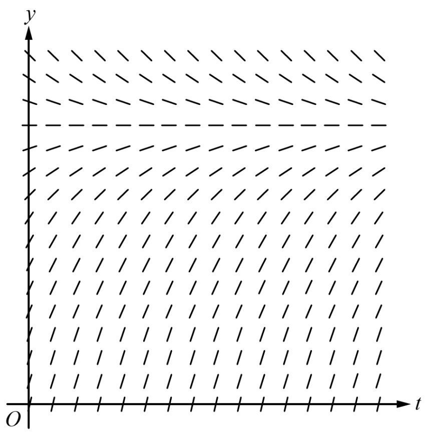
(c) Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of $G$ on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value $c$, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

. A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time $t$ hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are $0$ milligrams of the medication in the patient.

(a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.




(b) Using correct units, interpret the statement $\lim_{t \to \infty} A(t) = 12$ in the context of this problem.

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ with initial condition $A(0) = 0$.





(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time $t$ hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t + 2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$ ? Give a reason for your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

**STOP**

**END OF EXAM**

## CALCULUS AB
## SECTION II, Part A
**Time—30 minutes**
**Number of questions—2**

**A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.**

1. Fish enter a lake at a rate modeled by the function $E$ given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function $L$ given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and $t$ is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

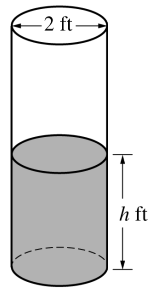
(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

(c) At what time $t$, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.








4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height $h$ of the water in the barrel with respect to time $t$ is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where $h$ is measured in feet and $t$ is measured in seconds. (The volume $V$ of a cylinder with radius $r$ and height $h$ is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for $h$ in terms of $t$.

AP CALCULUS AB • EXERCISE SHEET

# 5.7 The Second Derivative Test

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

**KEY WORDS** local extremum 局部极值 • local minimum 极小值 • critical point 临界点 • local maximum 极大值 • concave down 下凹

## Objective

Build the skills to answer exam questions on **the Second Derivative Test**.

**You must be able to:**

- use $f''$ at a critical point to classify a **local extremum 局部极值**

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

### Second Derivative Test

At a critical point $c$ where $f'(c) = 0$:

- $f''(c) > 0 \to$ local **minimum** (concave up);
- $f''(c) < 0 \to$ local **maximum** (concave down);
- $f''(c) = 0 \to$ **inconclusive**.

### Example

$f(x) = x^2 - 4x$: $f''(x) = 2 > 0$, so the critical point is a local minimum.

## 2 Practice

**2.1** State the Second Derivative Test for a local minimum. [1]

**2.2** For $f(x) = x^2 - 4x$ with $f'' = 2$, classify the critical point. [2]



**2.3** State when the test is inconclusive. [1]

## 3 Exam-style questions

**3.1** If $f'(c) = 0$ and $f''(c) < 0$, then $c$ is a [1]

- **A** local minimum
- **B** local maximum
- **C** inflection point
- **D** nothing

**3.2** If $f'(c) = 0$ and $f''(c) > 0$, then $c$ is a [1]

- **A** local maximum
- **B** local minimum
- **C** endpoint
- **D** asymptote

**3.3** $f(x) = x^2 - 6x$, with $f'(3) = 0$.

(a) Find $f''(x)$. [1]

(b) Evaluate $f''(3)$. [1]

(c) Classify $x = 3$. [1]

## Solutions

**2.1** if $f'(c) = 0$ and $f''(c) > 0$, then $c$ is a local minimum.

**2.2** $f'' = 2 > 0$, so it is a local minimum.

**2.3** when $f''(c) = 0$.

**3.1** **B**.

**3.2** **B**.

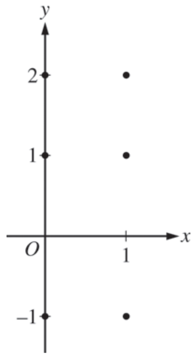
**3.3** (a) 2. (b) $2 > 0$. (c) local minimum.





4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.




(b) Find $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does $f$ have a relative minimum, a relative maximum, or neither at $x = 2$ ? Justify your answer.

(d) Find the values of the constants $m$ and $b$ for which $y = mx + b$ is a solution to the differential equation.

# 5.8 Sketching a Function and Its Derivatives

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

**KEY WORDS** increasing 递增 • concave up 上凹 • local minimum 极小值 • decreasing 递减

## Objective

Build the skills to answer exam questions on **sketching functions and their derivatives**.

**You must be able to:**

- relate features of $f$ to the sign and zeros of $f'$

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

**Sketching $f$ and $f'$**

- where $f$ has a local max/min, $f' = 0$ (crosses the axis);
- where $f$ is increasing, $f' > 0$ (above the axis);
- where $f$ is concave up, $f'$ is increasing.

Use these links to sketch one graph from the other.

## 2 Practice

**2.1** State the value of $f'$ at a local extremum of $f$. [1]

**2.2** State the sign of $f'$ where $f$ is increasing. [1]



**2.3** State what $f'$ does where $f$ is concave up. [2]

## 3 Exam-style questions

**3.1** At a local max of $f$, $f'$ is [1]

- **A** positive
- **B** zero
- **C** negative always
- **D** undefined

**3.2** Where $f$ is decreasing, its derivative $f'$ is [1]

- **A** positive
- **B** negative
- **C** zero
- **D** increasing

**3.3** $f$ has a local minimum at $x = 2$.

(a) State $f'(2)$. [1]

(b) State the sign of $f'$ just left of 2. [1]

(c) State the sign just right of 2. [1]

## Solutions

**2.1** $f' = 0$.

**2.2** positive ($f' > 0$).

**2.3** $f'$ is increasing.

**3.1** **B**.

**3.2** **B**.

**3.3** (a) 0. (b) negative. (c) positive.





# 5.9 Connecting f, f-prime, and f-double-prime

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

**KEY WORDS** local minimum 极小值 • concave up 上凹 • decreasing 递减 • increasing 递增

## Objective

Build the skills to answer exam questions on **connecting a function, its first derivative, and its second derivative**.

**You must be able to:**

- read $f$, $f'$, and $f''$ together to describe a graph

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

> **Connecting $f$, $f'$, $f''$**
> - $f'$ tells increasing/decreasing;
> - $f''$ tells concavity;
> - a local **min** of $f \Leftrightarrow f' = 0$ and $f'' > 0$;
> - an **inflection** point of $f \Leftrightarrow f''$ changes sign.
>
> **Example**
> $f(x) = x^3$: $f'(x) = 3x^2$ (zero at 0), $f''(x) = 6x$ (zero at 0).

## 2 Practice

**2.1** State what $f''$ tells you about $f$. [1]

**2.2** State the condition on $f'$ and $f''$ at a local minimum. [1]

**2.3** For $f(x) = x^3$, state where $f' = 0$ and where $f'' = 0$. [2]

## 3 Exam-style questions

**3.1** $f''$ gives information about a graph's [1]

- **A** slope
- **B** concavity
- **C** $x$-intercepts
- **D** domain

**3.2** A local minimum of $f$ satisfies [1]

- **A** $f' = 0$, $f'' > 0$
- **B** $f' = 0$, $f'' < 0$
- **C** $f'' = 0$ only
- **D** $f = 0$

**3.3** $f(x) = x^3$.

(a) Find $f'(x)$. [1]

(b) State where $f' = 0$. [1]

(c) State where $f'' = 0$. [1]



## Solutions

**2.1** its concavity (concave up where $f'' > 0$, down where $f'' < 0$).

**2.2** $f' = 0$ and $f'' > 0$.

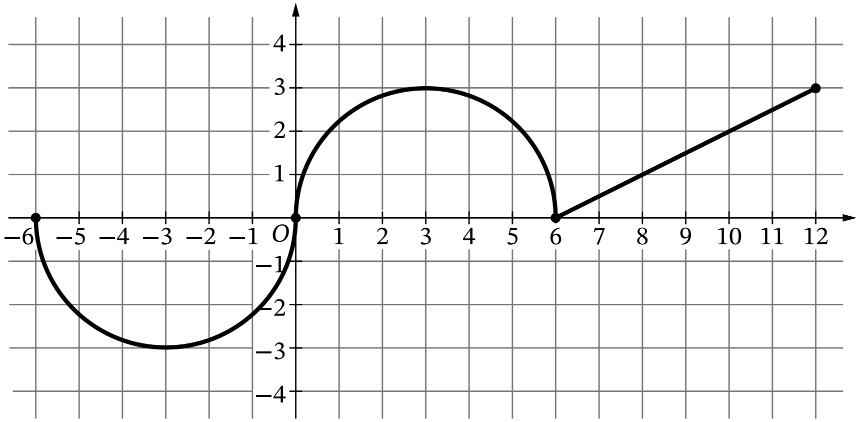
**2.3** $f' = 0$ at $x = 0$; $f'' = 0$ at $x = 0$.

**3.1** **B**.

**3.2** **A**.

**3.3** (a) $3x^2$. (b) $x = 0$. (c) $x = 0$.

**4.** The continuous function $f$ is defined on the closed interval $-6 \le x \le 12$. The graph of $f$, consisting of two semicircles and one line segment, is shown in the figure.




Graph of $f$

Let $g$ be the function defined by $g(x) = \int_6^x f(t)\,dt$.

**A.** Find $g'(8)$. Give a reason for your answer.

**B.** Find all values of $x$ in the open interval $-6 < x < 12$ at which the graph of $g$ has a point of inflection. Give a reason for your answer.

**C.** Find $g(12)$ and $g(0)$. Label your answers.

**D.** Find the value of $x$ at which $g$ attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.





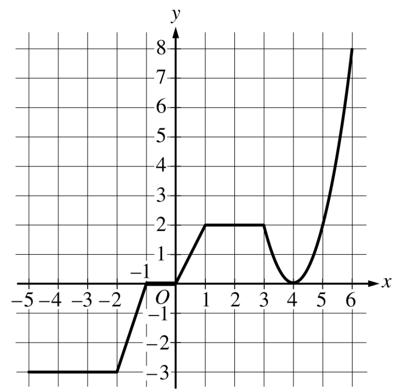
**CALCULUS AB**

**SECTION II, Part B**

**Time—1 hour**

**Number of questions—4**

**NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.**




Graph of $g$

3. The graph of the continuous function $g$, the derivative of the function $f$, is shown above. The function $g$ is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$ ?

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of $f$ both increasing and concave up? Give a reason for your answer.

(d) Find the $x$-coordinate of each point of inflection of the graph of $f$. Give a reason for your answer.

# 5.10 Introduction to Optimization

Name: ____________ Class: ________ Date: __________ **Total: 12 marks**

**KEY WORDS** constraint 约束 • optimization 最优化 • objective function 目标函数

## Objective

Build the skills to answer exam questions on **optimization 最优化** — turning a word problem into a function and finding its greatest or least value.

**You must be able to:**

- write a quantity to be maximized or minimized as a function of one variable (the **objective function 目标函数**)
- use a **constraint 约束** to eliminate the extra variable
- state the **domain** that makes physical sense for the problem

## 1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

### Naming the objective and the constraint

Optimization always has two ingredients: the **objective** (what you want biggest or smallest) and a **constraint** (a fixed condition). A farmer has 40 m of fence for a rectangular pen against a wall (no fence on the wall side).

- Objective: **area** $A = xy$ (maximize).
- Constraint: fence used $= x + 2y = 40$.

### Reducing to one variable

Solve the constraint for one variable and substitute. From $x + 2y = 40$, $x = 40 - 2y$, so

$$A(y) = (40 - 2y)\,y = 40y - 2y^2.$$

Now $A$ depends on the single variable $y$ — ready to differentiate.

### Stating a sensible domain

Lengths are positive, so $y > 0$ and $x = 40 - 2y > 0$, giving $0 < y < 20$. The optimization only searches this **domain**; an answer outside it is rejected.



### Reading what the question asks for

Check whether the question wants the **variable** (the $y$ that optimizes), the **maximum value** ($A$ itself), or both. A common lost mark is giving $y$ when the question asked for the area.

## 2 Practice

Now apply the methods above.

**2.1** A rectangle has perimeter 24 cm. Write its **area** as a function of its width $x$. [2]

**2.2** An open-top box has a square base of side $x$ and volume 32 cm$^3$. Write its **surface area** (base + 4 sides) as a function of $x$. [2]

**2.3** For the farmer's pen $A(y) = 40y - 2y^2$, state the domain of $y$. [1]

## 3 Exam-style questions

**3.1** A quantity $Q$ is to be maximized subject to a constraint. What is the **first** step? [1]

- **A** differentiate $Q$ immediately
- **B** write $Q$ as a function of a single variable using the constraint
- **C** set $Q = 0$
- **D** find the second derivative

**3.2** A cylindrical can has volume 355 $\text{cm}^3$. Its surface area is $S = 2\pi r^2 + 2\pi rh$.

(a) Use the volume constraint $\pi r^2 h = 355$ to write $S$ as a function of $r$ only. [2]

(b) State the domain of $r$. [1]

**3.3** A 600 $\text{cm}^2$ sheet of card is folded into an open box with a square base of side $x$ and height $h$, using all the card ($x^2 + 4xh = 600$). Write the **volume** $V$ as a function of $x$ alone. [3]



## Solutions

**2.1** Perimeter $2x + 2w = 24 \Rightarrow$ length $= 12 - x$; $A = x(12 - x) = 12x - x^2$.

**2.2** Height from $x^2h = 32$, so $h = 32/x^2$; $S = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$.

**2.3** $0 < y < 20$.

**3.1 B** — reduce to a single variable using the constraint before differentiating.

**3.2** (a) $h = \frac{355}{\pi r^2}$; $S(r) = 2\pi r^2 + 2\pi r\left(\frac{355}{\pi r^2}\right) = 2\pi r^2 + \frac{710}{r}$. (b) $r > 0$.

**3.3** $h = \frac{600 - x^2}{4x}$; $V = x^2h = x^2\left(\frac{600 - x^2}{4x}\right) = \frac{600x - x^3}{4} = 150x - \frac{1}{4}x^3$.

AP CALCULUS AB • EXERCISE SHEET

# 5.11 Solving Optimization Problems

Name: ____________ Class: ________ Date: __________ **Total: 17 marks**

**KEY WORDS** critical points 临界点 • objective function 目标函数 • local maximum 极大值 • optimization 最优化 • point of inflection 拐点

## Objective

Build the skills to answer exam questions on **solving optimization problems** — carrying the setup through to a justified maximum or minimum.

**You must be able to:**

- differentiate the **objective function** 目标函数 and solve $f'(x) = 0$ for the **critical points** 临界点
- **justify** that a critical point is a maximum or minimum (sign of $f'$, or the second-derivative test, or the closed-interval candidates)
- give the answer the question asks for, with units

## 1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

### The full optimization method

Maximize the pen area $A(y) = 40y - 2y^2$ on $0 < y < 20$.

1. Differentiate: $A'(y) = 40 - 4y$.
2. Solve $A'(y) = 0$: $y = 10$.
3. Justify: $A''(y) = -4 < 0$, so $y = 10$ gives a **maximum**.
4. Answer: $x = 40 - 2(10) = 20$, so $A = 20 \times 10 = 200$ m$^2$.

### Justifying with the first-derivative sign

If the second derivative is awkward, test the sign of $f'$ on each side of the critical point. Here $A'(9) = +4 > 0$ and $A'(11) = -4 < 0$: $f$ rises then falls, so the critical point is a **maximum**. A justification is **required** for full marks.



### Minimizing a cost

A can's surface area is $S(r) = 2\pi r^2 + \dfrac{710}{r}$. Then $S'(r) = 4\pi r - \dfrac{710}{r^2}$. Setting $S' = 0$: $4\pi r^3 = 710$, so $r = \left(\dfrac{710}{4\pi}\right)^{1/3} \approx 3.84$ cm. Since $S''(r) = 4\pi + \dfrac{1420}{r^3} > 0$, this is a **minimum**.

## 2 Practice

Now apply the methods above.

**2.1** For $A(y) = 40y - 2y^2$, find $A'(y)$ and solve $A'(y) = 0$. [2]

**2.2** The area function is $A = 12x - x^2$. Find the width $x$ that maximizes the area, and **justify** it is a maximum. [3]

**2.3** A function $C(x) = x^2 + \dfrac{16}{x}$ models a cost for $x > 0$. Find the $x$ that minimizes $C$. [3]

## 3 Exam-style questions

**3.1** A critical point of $f$ has $f'(c) = 0$ and $f''(c) < 0$. The point is a [1]

- **A** local minimum
- **B** local maximum
- **C** point of inflection
- **D** endpoint

**3.2** A rectangular field of area $A = x(200 - x)$ m$^2$ is enclosed, where $0 < x < 200$.

(a) Find the value of $x$ that maximizes $A$. [2]

(b) Justify that your value gives a maximum. [1]

(c) State the maximum area. [1]

**3.3** An open box is made from card, with volume $V(x) = 150x - \frac{1}{4}x^3$ for $0 < x < \sqrt{600}$.

(a) Show that $V'(x) = 150 - \frac{3}{4}x^2$. [1]

(b) Find the value of $x$ that maximizes the volume, and justify it. [3]



## Solutions

**2.1** $A'(y) = 40 - 4y$; $40 - 4y = 0 \Rightarrow y = 10$.

**2.2** $A'(x) = 12 - 2x = 0 \Rightarrow x = 6$; $A''(x) = -2 < 0$, so $x = 6$ is a maximum.

**2.3** $C'(x) = 2x - \dfrac{16}{x^2} = 0$; $2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x = 2$.

**3.1 B** — $f'(c) = 0$ with $f''(c) < 0$ is a local maximum.

**3.2** (a) $A = 200x - x^2$, $A'(x) = 200 - 2x = 0 \Rightarrow x = 100$. (b) $A''(x) = -2 < 0$, so it is a maximum. (c) $A = 100(100) = 10\,000$ m$^2$.

**3.3** (a) $V'(x) = 150 - \frac{3}{4}x^2$. (b) $150 - \frac{3}{4}x^2 = 0 \Rightarrow x^2 = 200 \Rightarrow x = \sqrt{200} \approx 14.1$; $V''(x) = -\frac{3}{2}x < 0$ for $x > 0$, so it is a maximum.

AP CALCULUS AB • EXERCISE SHEET

# 5.12 Behaviors of Implicit Relations

Name: ____________ Class: ________ Date: __________ **Total: 10 marks**

**KEY WORDS** horizontal tangent 水平切线 • vertical tangent 垂直切线 • concavity 凹凸性 • concave down 下凹 • concave up 上凹

## Objective

Build the skills to answer exam questions on the **behavior of implicit relations** — using $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ found by implicit differentiation to locate features of a curve.

**You must be able to:**

- find where an implicit curve has a **horizontal tangent** 水平切线 ($\frac{dy}{dx} = 0$) or a **vertical tangent** 垂直切线 ($\frac{dy}{dx}$ undefined)
- decide **concavity** 凹凸性 from the sign of $\frac{d^2y}{dx^2}$ on the curve
- evaluate these using a point that lies **on** the curve

## 1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

### Horizontal tangents

For $x^2 + y^2 = 25$, implicit differentiation gives $2x + 2y\frac{dy}{dx} = 0$, so

$$\frac{dy}{dx} = -\frac{x}{y}.$$

A **horizontal** tangent needs $\frac{dy}{dx} = 0$, i.e. the numerator $x = 0$ (with $y \neq 0$): the points $(0, 5)$ and $(0, -5)$.

### Vertical tangents

A **vertical** tangent occurs where $\frac{dy}{dx}$ is undefined — the denominator zero, $y = 0$ (with $x \neq 0$): the points $(5, 0)$ and $(-5, 0)$. This matches the circle's shape.



### Concavity on an implicit curve

Differentiate $\frac{dy}{dx} = -\frac{x}{y}$ again (quotient + chain rule):

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} = -\frac{y - x(-x/y)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}.$$

Where $y > 0$ this is negative (**concave down**, the top of the circle); where $y < 0$ it is positive (**concave up**).

## 2 Practice

Now apply the methods above.

**2.1** For $x^2 + y^2 = 25$, state $\frac{dy}{dx}$. [1]

**2.2** Find the point(s) on $x^2 + y^2 = 25$ where the tangent is **horizontal**. [2]

**2.3** A curve satisfies $\frac{dy}{dx} = \frac{2x}{y}$. State where its tangent line is **vertical**. [1]

## 3 Exam-style questions

**3.1** A curve has $\frac{dy}{dx} = \frac{x-1}{y}$. A horizontal tangent occurs where [1]

- **A** $y = 0$
- **B** $x = 0$
- **C** $x = 1$
- **D** $x = y$

**3.2** The curve $x^2 + xy + y^2 = 7$ passes through $(1, 2)$. Its derivative is $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$.

(a) Find the slope of the tangent at $(1, 2)$. [2]

(b) Determine whether the tangent at $(1, 2)$ is horizontal, vertical, or neither. [1]

**3.3** For $x^2 + y^2 = 25$ it can be shown that $\dfrac{d^2y}{dx^2} = -\dfrac{25}{y^3}$. State, with a reason, whether the curve is concave up or concave down at the point $(3, 4)$. [2]



## Solutions

**2.1** $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

**2.2** Set numerator $x = 0$; points $(0, 5)$ and $(0, -5)$.

**2.3** Vertical where denominator $y = 0$, i.e. at $(\pm 5, 0)$ — where the curve meets the $x$-axis.

**3.1** **C** — a horizontal tangent needs the numerator zero, $x - 1 = 0$, so $x = 1$.

**3.2** (a) $\dfrac{dy}{dx} = -\dfrac{2(1) + 2}{1 + 2(2)} = -\dfrac{4}{5}$. (b) Neither — the slope $-\frac{4}{5}$ is finite and non-zero.

**3.3** $\dfrac{d^2y}{dx^2} = -\dfrac{25}{4^3} = -\dfrac{25}{64} < 0$; negative, so the curve is **concave down** at $(3, 4)$.

**6.** Consider the curve $G$ defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

**A.** Show that $\dfrac{dy}{dx} = \dfrac{-x}{2\left(3y^2 - 2y - 1\right)}$.

**B.** There is a point $P$ on the curve $G$ near $(2, -1)$ with $x$-coordinate 1.6. Use the line tangent to the curve at $(2, -1)$ to approximate the $y$-coordinate of point $P$.

**C.** For $x > 0$ and $y > 0$, there is a point $S$ on the curve $G$ at which the line tangent to the curve at that point is vertical. Find the $y$-coordinate of point $S$. Show the work that leads to your answer.

**D.** A particle moves along the curve $H$ defined by the equation $2xy + \ln y = 8$. At the instant when the particle is at the point $(4, 1)$, $\dfrac{dx}{dt} = 3$. Find $\dfrac{dy}{dt}$ at that instant. Show the work that leads to your answer.

**STOP**

**END OF EXAM**



5. Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\dfrac{dy}{dx} = \dfrac{-2x}{3 + 4y}$.

(a) There is a point on the curve near $(2, 4)$ with $x$-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the $y$-coordinate of this point.

(b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

(c) The curve intersects the positive $x$-axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

(d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the $y$-coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the $x$-coordinate of the particle's position with respect to time?

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**





6. Consider the curve given by the equation $6xy = 2 + y^3$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2y}{y^2 - 2x}$.

(b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\dfrac{dx}{dt} = \dfrac{2}{3}$ unit per second. What is the value of $\dfrac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

**STOP**

**END OF EXAM**

6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\dfrac{dy}{dx} = \dfrac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\dfrac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

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**STOP**

**END OF EXAM**