Week 10

AP Calculus AB

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

3. Differentiation: Composite, Implicit, and Inverse Functions Starter Quiz · 课前小测

AP Calculus AB Name: ______________ Score: ______

1. The Chain Rule gives $\frac{d}{dx}[f(g(x))] =$

   - ☐ $f'(x) \cdot g'(x)$
   - ☐ $f'(g(x)) \cdot g'(x)$
   - ☐ $f'(g(x))$
   - ☐ $f(g'(x))$

2. Differentiating with respect to x, $\frac{d}{dx}[y^2] =$

   - ☐ $2y$
   - ☐ $2y\frac{dy}{dx}$
   - ☐ $2x$
   - ☐ 2

3. The derivative of an inverse is $\left(f^{-1}\right)'(x) =$

   - ☐ $f'(x)$
   - ☐ $\frac{1}{f'(f^{-1}(x))}$
   - ☐ $\frac{1}{f'(x)}$
   - ☐ $-f'(x)$

4. What is $\frac{d}{dx}[\arcsin x]$?

   - ☐ $\frac{1}{1+x^2}$
   - ☐ $\frac{1}{\sqrt{1-x^2}}$
   - ☐ $\cos x$
   - ☐ $-\frac{1}{\sqrt{1-x^2}}$

5. Which rule does $y = x^2 \sin x$ need at the top level?

   - ☐ quotient rule
   - ☐ product rule

☐ chain rule only

☐ power rule only

6. For $x^2 + y^2 = 25$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Find the slope at $(3, 4)$ (a decimal).

____

7. If $f(2) = 5$ and $f'(2) = 4$, find $\left(f^{-1}\right)'(5)$.

____

8. For $f(x) = x^3 - 2x^2 + 5x$, $f'(x) = 6x - 4$. Find $f'(2)$.

____

9. For $y = (2x+1)^3$, find y' at $x = 0$. (Chain rule: $3(2x+1)^2 \cdot 2$.)

____

10. The factor $\frac{dy}{dx}$ is added to every term, including pure x terms like x^2.

☐ True ☐ False

3. Differentiation: Composite, Implicit, and Inverse Functions Answers · 答案

AP Calculus AB

1. $f'(g(x)) \cdot g'(x)$

   Outer derivative at the inner, times the inner derivative.

2. $2y\dfrac{dy}{dx}$

   Chain rule: $2y$ times the inner derivative $\frac{dy}{dx}$.

3. $\dfrac{1}{f'(f^{-1}(x))}$

   One over the original derivative at the matching input $f^{-1}(x)$.

4. $\dfrac{1}{\sqrt{1-x^2}}$

   arcsin has the root form $\frac{1}{\sqrt{1-x^2}}$.

5. **product rule**

   It is a product of x^2 and $\sin x \to$ product rule.

6. **-0.75**

   $-\frac{3}{4} = -0.75$.

7. **0.25**

   $f^{-1}(5) = 2$, so $\left(f^{-1}\right)'(5) = \frac{1}{f'(2)} = \frac{1}{4}$.

8. **8**

   $f'(2) = 6(2) - 4 = 8$.

9. **6**

   $y' = 6(2x+1)^2$; at 0: $6(1) = 6$.

10. **False**

    Only terms containing y get the $\frac{dy}{dx}$ factor.

AP CALCULUS AB • EXERCISE SHEET

3.1 The Chain Rule

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS chain rule 链式法则 • composite function 复合函数

Objective

Build the skills to answer exam questions on **the chain rule**.

You must be able to:

- differentiate a **composite function 复合函数** with the **chain rule 链式法则**

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **The chain rule**

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

— derivative of the outer (evaluated at the inner) times the derivative of the inner.

■ **Example**

$\frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot 2x = 2x\cos(x^2)$.

2 Practice

2.1 State the chain rule. [1]

2.2 Differentiate $(x^2 + 1)^3$. [2]

2.3 State what the chain rule is for. [1]

3 Exam-style questions

3.1 $\frac{d}{dx}[f(g(x))]$ equals [1]

- **A** $f'(x)g'(x)$
- **B** $f'(g(x)) \cdot g'(x)$
- **C** $f'(g(x))$
- **D** $g'(f(x))$

3.2 $\frac{d}{dx}[\sin(x^2)]$ equals [1]

- **A** $\cos(x^2)$
- **B** $2x\cos(x^2)$
- **C** $2x\sin(x^2)$
- **D** $\cos(2x)$

3.3 $y = (3x+1)^4$.

(a) Identify the outer and inner functions. [1]

(b) Apply the chain rule. [1]

(c) State y'. [1]

Solutions

2.1 $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$.

2.2 $3(x^2+1)^2 \cdot 2x = 6x(x^2+1)^2$.

2.3 differentiating a composite (a function of a function).

3.1 **B**.

3.2 **B**.

3.3 (a) outer u^4, inner $3x+1$. (b) $4(3x+1)^3 \cdot 3$. (c) $y' = 12(3x+1)^3$.

6. The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	0	2	3	6
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Part A

Find $\lim\limits_{x \to 2} \dfrac{f(x)}{x}$, or state that the limit does not exist.

Part B

Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

Part C

Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

Part D

Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\,dt$.

i. Find $k'(x)$.

ii. Find $k''(3)$. Show the work that leads to your answer.

STOP

END OF EXAM

5. Two particles, H and J, are moving along the x-axis. For $0 \le t \le 5$, the position of particle H at time t is given by $x_H(t) = e^{t^2 - 4t}$ and the velocity of particle J at time t is given by $v_J(t) = 2t(t^2 - 1)^3$.

A. Find the velocity of particle H at time $t = 1$. Show the work that leads to your answer.

B. During what open intervals of time t, for $0 < t < 5$, are particles H and J moving in opposite directions? Give a reason for your answer.

C. It can be shown that $v_J'(2) > 0$. Is the speed of particle J increasing, decreasing, or neither at time $t = 2$? Give a reason for your answer.

D. Particle J is at position $x = 7$ at time $t = 0$. Find the position of particle J at time $t = 2$. Show the work that leads to your answer.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

5. The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

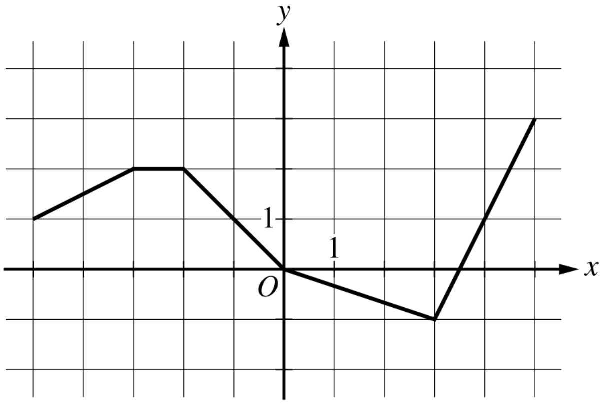
(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

x	$g(x)$	$g'(x)$
–5	10	–3
–4	5	–1
–3	2	4
–2	3	1
–1	1	–2
0	0	–3



Graph of h

6. Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x.

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

(d) Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.

STOP

END OF EXAM

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	–6	3	2	8
2	2	–2	–3	0
3	8	7	6	2
6	4	5	3	–1

6. The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

   (a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

   (b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

   (c) Evaluate $\int_1^3 f''(2x)\, dx$.

STOP

END OF EXAM

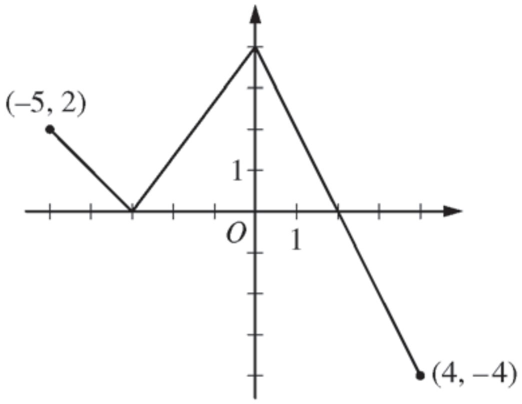
CALCULUS AB

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\, dt$.

   (a) Find $g(3)$.

   (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

   (c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

   (d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

   (a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

   (b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

   (c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

   (d) Evaluate $\int_{-2}^{3} f'(g(x))\,g'(x)\,dx$.

3.2 Implicit Differentiation

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS implicit differentiation 隐函数求导 • implicit relation 隐式关系 • chain rule 链式法则

Objective

Build the skills to answer exam questions on **implicit differentiation**.

You must be able to:

- differentiate an **implicit relation 隐式关系** by attaching $\dfrac{dy}{dx}$ to every y
- solve for $\dfrac{dy}{dx}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Implicit differentiation

Differentiate both sides with respect to x, writing y' each time you differentiate y (the chain rule), then solve for y'.

■ Example

$x^2 + y^2 = 25 \;\Rightarrow\; 2x + 2y\,y' = 0 \;\Rightarrow\; y' = -\dfrac{x}{y}.$

2 Practice

2.1 State what to attach when differentiating y. [1]

2.2 Differentiate $x^2 + y^2 = 25$ implicitly. [2]

2.3 State the final step. [1]

3 Exam-style questions

3.1 In implicit differentiation, differentiating y^2 gives [1]

- **A** $2y$
- **B** $2y\,y'$
- **C** y'
- **D** $2x$

3.2 After differentiating, you [1]

- **A** stop
- **B** solve for $\dfrac{dy}{dx}$
- **C** set $y = 0$
- **D** integrate

3.3 $x^2 + y^2 = 25$.

(a) Differentiate both sides. [1]

(b) Isolate the y' term. [1]

(c) Solve for y'. [1]

Solutions

2.1 the factor $\frac{dy}{dx}$ (i.e. y').

2.2 $2x + 2y\,y' = 0$, so $y' = -\frac{x}{y}$.

2.3 solve the equation for $\frac{dy}{dx}$.

3.1 **B**.

3.2 **B**.

3.3 (a) $2x + 2y\,y' = 0$. (b) $2y\,y' = -2x$. (c) $y' = -\frac{x}{y}$.

6. Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

A. Show that $\frac{dy}{dx} = \frac{-x}{2\left(3y^2 - 2y - 1\right)}$.

B. There is a point P on the curve G near $(2, -1)$ with x-coordinate 1.6. Use the line tangent to the curve at $(2, -1)$ to approximate the y-coordinate of point P.

C. For $x > 0$ and $y > 0$, there is a point S on the curve G at which the line tangent to the curve at that point is vertical. Find the y-coordinate of point S. Show the work that leads to your answer.

D. A particle moves along the curve H defined by the equation $2xy + \ln y = 8$. At the instant when the particle is at the point $(4, 1)$, $\frac{dx}{dt} = 3$. Find $\frac{dy}{dt}$ at that instant. Show the work that leads to your answer.

STOP

END OF EXAM

5. Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$.

(a) There is a point on the curve near $(2, 4)$ with x-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y-coordinate of this point.

(b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

(c) The curve intersects the positive x-axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

(d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y-coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x-coordinate of the particle's position with respect to time?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

6. Consider the curve given by the equation $6xy = 2 + y^3$.

   (a) Show that $\dfrac{dy}{dx} = \dfrac{2y}{y^2 - 2x}$.

   (b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

   (c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

   (d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\dfrac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\dfrac{dx}{dt} = \dfrac{2}{3}$ unit per second. What is the value of $\dfrac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

5. Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y \cos x}{4y - \sin x}$.

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\dfrac{dy}{dx} = \dfrac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\dfrac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

STOP

END OF EXAM

3.3 Differentiating Inverse Functions

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS reciprocal 倒数 • inverse function 反函数

Objective

Build the skills to answer exam questions on **differentiating inverse functions**.

You must be able to:

- apply $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

> ■ **Derivative of an inverse**
>
> $$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$
>
> The slope of the **inverse function 反函数** is the reciprocal of the slope of f at the corresponding point.
>
> ■ **Example**
>
> If $f(2) = 5$ and $f'(2) = 4$, then $(f^{-1})'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{4}$.

2 Practice

2.1 State the inverse-derivative formula. [1]

2.2 If $f'(2) = 4$ and $f(2) = 5$, find $(f^{-1})'(5)$. [2]

2.3 State the relationship between the two slopes. [1]

3 Exam-style questions

3.1 $(f^{-1})'(a)$ equals [1]

- **A** $f'(a)$
- **B** $\dfrac{1}{f'(f^{-1}(a))}$
- **C** $f'(f^{-1}(a))$
- **D** $\dfrac{1}{a}$

3.2 The slope of f^{-1} is the __________ of the slope of f. [1]

- **A** same
- **B** reciprocal
- **C** negative
- **D** square

3.3 $f(3) = 7$, $f'(3) = 2$.

(a) Find $f^{-1}(7)$. [1]

(b) Apply the formula. [1]

(c) Find $(f^{-1})'(7)$. [1]

Solutions

2.1 $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$.

2.2 $f^{-1}(5) = 2$, so $(f^{-1})'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{4}$.

2.3 they are reciprocals of each other.

3.1 **B**.

3.2 **B**.

3.3 (a) 3. (b) $\dfrac{1}{f'(3)}$. (c) $\dfrac{1}{2}$.

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3.4 Differentiating Inverse Trigonometric Functions

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS inverse trigonometric functions 反三角函数

Objective

Build the skills to answer exam questions on **differentiating inverse trigonometric functions**.

You must be able to:

- state and use the derivatives of $\arcsin x$, $\arccos x$, and $\arctan x$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Inverse trig derivatives**

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}.$$

2 Practice

2.1 State $\frac{d}{dx}[\arcsin x]$. [1]

2.2 State $\frac{d}{dx}[\arctan x]$. [1]

2.3 State $\frac{d}{dx}[\arccos x]$. [2]

3 Exam-style questions

3.1 $\frac{d}{dx}[\arctan x]$ equals [1]

- **A** $\frac{1}{\sqrt{1-x^2}}$
- **B** $\frac{1}{1+x^2}$
- **C** $-\frac{1}{1+x^2}$
- **D** $\frac{1}{x}$

3.2 $\frac{d}{dx}[\arcsin x]$ equals [1]

- **A** $\frac{1}{1+x^2}$
- **B** $\frac{1}{\sqrt{1-x^2}}$
- **C** $-\frac{1}{\sqrt{1-x^2}}$
- **D** $\arccos x$

3.3 Differentiate each.

(a) $\arcsin x$. [1]

(b) $\arctan x$. [1]

(c) $\arccos x$. [1]

Solutions

2.1 $\dfrac{1}{\sqrt{1-x^2}}$.

2.2 $\dfrac{1}{1+x^2}$.

2.3 $-\dfrac{1}{\sqrt{1-x^2}}$.

3.1 **B**.

3.2 **B**.

3.3 (a) $\dfrac{1}{\sqrt{1-x^2}}$. (b) $\dfrac{1}{1+x^2}$. (c) $-\dfrac{1}{\sqrt{1-x^2}}$.

3.5 Selecting Procedures for Calculating Derivatives

Name: ______ Class: ______ Date: ______ **Total: 9 marks**

KEY WORDS chain rule 链式法则

Objective

Build the skills to answer exam questions on **selecting procedures for calculating derivatives**.

You must be able to:

- recognise the structure of an expression and pick the right rule (power, product, quotient, chain)

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Choosing a rule

- a power of x → **power rule**;
- a product → **product rule**;
- a quotient → **quotient rule**;
- a composite (function of a function) → **chain rule**.

Combine them when an expression mixes structures.

2 Practice

2.1 State which rule to use for a composite. [1]

2.2 State which rule to use for a product. [1]

2.3 For $y = x^2 \sin x$, name the rule. [2]

3 Exam-style questions

3.1 For $y = (x^2 + 1)^5$, use the [1]

- **A** quotient rule
- **B** chain rule
- **C** product rule only
- **D** no rule

3.2 For $y = \dfrac{x}{x+1}$, use the [1]

- **A** power rule
- **B** quotient rule
- **C** chain rule
- **D** sum rule

3.3 Name the rule for each.

(a) $y = x^3$. [1]

(b) $y = x\,e^x$. [1]

(c) $y = \sin(x^2)$. [1]

Solutions

2.1 the chain rule.

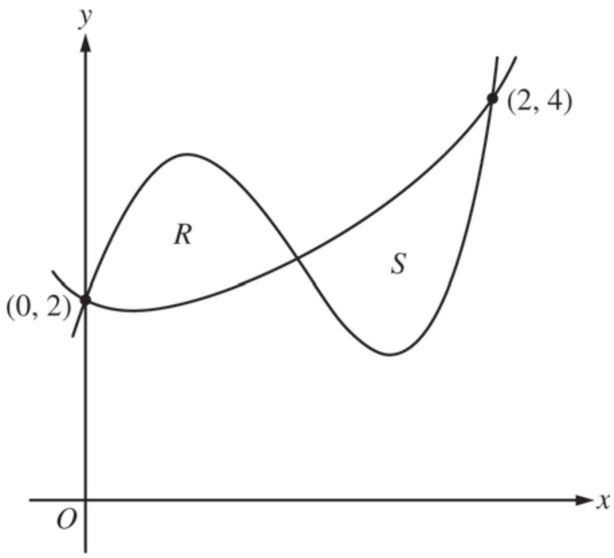
2.2 the product rule.

2.3 the product rule.

3.1 **B**.

3.2 **B**.

3.3 (a) power rule. (b) product rule. (c) chain rule.



2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2 - 2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

   (a) Find the sum of the areas of regions R and S.

   (b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

   (c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

END OF PART A OF SECTION II

3.6 Calculating Higher-Order Derivatives

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS second derivative 二阶导数

Objective

Build the skills to answer exam questions on **higher-order derivatives**.

You must be able to:

- find the **second derivative 二阶导数** f''
- use the notation $\frac{d^2y}{dx^2}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Higher-order derivatives

The second derivative $f'' = (f')'$ is the derivative of the first derivative. Notation: $f''(x)$, $\frac{d^2y}{dx^2}$.

■ Example

$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$.

2 Practice

2.1 State what the second derivative is. [1]

2.2 For $f(x) = x^3$, find f''. [2]

2.3 State the notation for the second derivative. [1]

3 Exam-style questions

3.1 The second derivative is the derivative of [1]

- **A** f
- **B** f'
- **C** the integral
- **D** a constant

3.2 $\dfrac{d^2y}{dx^2}$ denotes the [1]

- **A** first derivative
- **B** second derivative
- **C** integral
- **D** slope only

3.3 $f(x) = x^4$.

(a) Find f'. [1]

(b) Find f''. [1]

(c) State $f''(1)$. [1]

Solutions

2.1 the derivative of the first derivative.

2.2 $f' = 3x^2$, so $f'' = 6x$.

2.3 $f''(x)$ or $\dfrac{d^2y}{dx^2}$.

3.1 **B**.

3.2 **B**.

3.3 (a) $4x^3$. (b) $12x^2$. (c) 12.

6. The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	0	2	3	6
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Part A

Find $\lim_{x\to 2} \frac{f(x)}{x}$, or state that the limit does not exist.

Part B

Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

Part C

Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

Part D

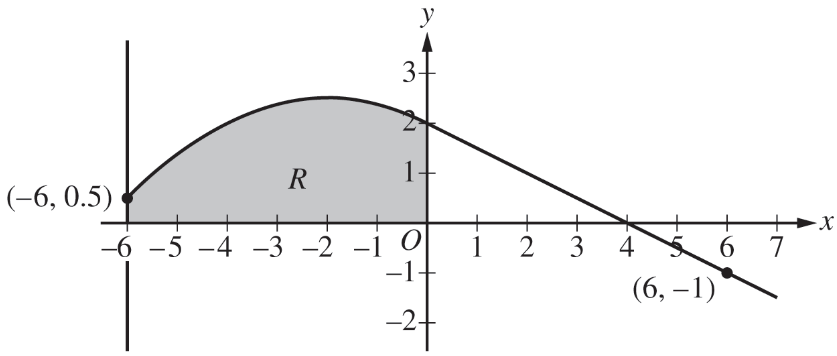
Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\,dt$.

i. Find $k'(x)$.

ii. Find $k''(3)$. Show the work that leads to your answer.

STOP

END OF EXAM



Graph of f

4. The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\, dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

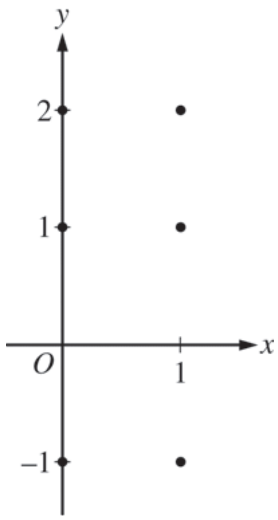
(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\, dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

   (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



   (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

   (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

   (d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

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6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

   (a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

   (b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

   (c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

STOP

END OF EXAM