Week 1

AP Calculus AB

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

1. Limits and Continuity

Starter Quiz · 课前小测

AP Calculus AB Name: ____________ Score: ______

1. The slope of a **secant line** between two points on a curve gives the...
   - ☐ instantaneous rate of change
   - ☐ average rate of change over that interval
   - ☐ exact value of the function
   - ☐ area under the curve
2. The notation $\lim_{x\to c} f(x) = L$ means that as x...
   - ☐ equals c, the output is exactly L
   - ☐ gets close to c (without equalling it), the output heads toward L
   - ☐ grows without bound, $f(x)$ reaches L
   - ☐ is greater than c, $f(x)$ equals L
3. To read $\lim_{x\to c} f(x)$ from a graph, you should look at...
   - ☐ the plotted dot exactly at $x = c$
   - ☐ the height the curve heads toward from both sides of c
   - ☐ the y-intercept of the graph
   - ☐ the highest point on the whole graph
4. A table gives $f(2.9) = 6.9$, $f(2.99) = 6.99$, $f(3.01) = 7.01$, $f(3.1) = 7.1$. Estimate $\lim_{x\to 3} f(x)$.

   ____
5. Evaluate $\lim\limits_{x\to 2}(3x^2 - x + 4)$.

   ____
6. Evaluate $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4}$.

   ____
7. Direct substitution into $\lim_{x\to 4} \dfrac{x-4}{x^2-16}$ gives $\frac{0}{0}$. What next?
   - ☐ report the limit as 0
   - ☐ factor and cancel the common $(x-4)$
   - ☐ declare it does not exist
   - ☐ multiply by a conjugate

8. To apply the Squeeze Theorem to f near c, you need **all** of...
   - ☐ bounds $g \le f \le h$ holding near c
   - ☐ the two bounds to have equal limits at c
   - ☐ f to be a polynomial
   - ☐ the inequality to hold on an interval around c

   Tick every correct option.
9. A graph shows the curve heading to height 3 from both sides of $x = 2$, with a filled dot at $(2, 7)$. Which statement matches?
   - ☐ $\lim_{x\to 2} f(x) = 7$
   - ☐ $\lim_{x\to 2} f(x) = 3$ and $f(2) = 7$
   - ☐ the limit does not exist
   - ☐ $f(2) = 3$
10. Select **all** conditions required for f to be continuous at c.
    - ☐ $f(c)$ is defined
    - ☐ $\lim_{x\to c} f(x)$ exists
    - ☐ $\lim_{x\to c} f(x) = f(c)$
    - ☐ f is a polynomial

    Tick every correct option.

AP Calculus AB

1. **average rate of change over that interval**

   A secant joins two points, so its slope is the **average rate of change** over that interval.

2. **gets close to c (without equalling it), the output heads toward L**

   $x \to c$ means x **approaches** c without touching it, and $f(x)$ heads toward L.

3. **the height the curve heads toward from both sides of c**

   A limit is the shared height approached from both sides — the dot at c is irrelevant.

4. **7**

   Both sides close in on 7, so the limit is 7.

5. **14**

   Direct substitution: $3(4) - 2 + 4 = 14$.

6. **8**

   Factor: $\frac{(x-4)(x+4)}{x-4} = x + 4 \to 8$.

7. **factor and cancel the common $(x - 4)$**

   Polynomial over polynomial, both zero at $4 \to$ factor: $\frac{x-4}{(x-4)(x+4)} = \frac{1}{x+4} \to \frac{1}{8}$.

8. **bounds $g \le f \le h$ holding near c; the two bounds to have equal limits at c; the inequality to hold on an interval around c**

   You need two-sided bounds with equal limits, valid near c. f need not be a polynomial — that is the whole point.

9. $\lim_{x\to 2} f(x) = 3$ **and** $f(2) = 7$

   The curve's approach gives the limit 3; the filled dot gives the separate value $f(2) = 7$.

10. $f(c)$ **is defined;** $\lim_{x\to c} f(x)$ **exists;** $\lim_{x\to c} f(x) = f(c)$

    The three conditions are defined, limit-exists, and equal. Being a polynomial is not required.

AP CALCULUS AB • EXERCISE SHEET

1.1 Introducing Calculus: Can Change Occur at an Instant?

Name: ____________ Class: __________ Date: __________ **Total: 9 marks**

KEY WORDS change 变化 • average rate of change 平均变化率 • interval 区间 • instantaneous rate of change 瞬时变化率 • limit 极限

Objective

Build the skills to answer exam questions on **whether change can occur at an instant**.

You must be able to:

- find an **average rate of change** 平均变化率 over an interval
- describe the **instantaneous rate of change** 瞬时变化率 as a limit of average rates

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Average vs instantaneous rate

The **average rate of change** of f over $[a, b]$ is $\dfrac{f(b) - f(a)}{b - a}$.

The **instantaneous rate** at a point is the **limit** of these average rates as the interval shrinks to zero — this is the idea calculus makes precise.

■ Example

$f(x) = x^2$ on $[1, 3]$: average rate $= \dfrac{9 - 1}{3 - 1} = 4$.

2 Practice

2.1 State the average rate of change of f on $[a, b]$. [1]

2.2 State what the instantaneous rate of change is the limit of. [1]

2.3 For $f(x) = x^2$, find the average rate of change on $[1, 3]$. [2]

3 Exam-style questions

3.1 The instantaneous rate of change is [1]

- **A** the average over $[a, b]$
- **B** the limit of average rates as the interval shrinks
- **C** always zero
- **D** the slope of a single secant line

3.2 The average rate of change of f on $[a, b]$ is [1]

- **A** $f(b) \cdot f(a)$
- **B** $\dfrac{f(b) - f(a)}{b - a}$
- **C** $f(b) + f(a)$
- **D** $b - a$

3.3 $f(x) = x^2$.

(a) Find $f(1)$ and $f(3)$. [1]

(b) Find the average rate of change on $[1, 3]$. [1]

(c) State what a shrinking interval approaches. [1]

Solutions

2.1 $\dfrac{f(b) - f(a)}{b - a}$.

2.2 the average rates of change over intervals shrinking to zero.

2.3 $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{9 - 1}{2} = 4$.

3.1 B.

3.2 B.

3.3 (a) $f(1) = 1$, $f(3) = 9$. (b) $\dfrac{9 - 1}{2} = 4$. (c) the instantaneous rate of change.

AP CALCULUS AB • EXERCISE SHEET

1.2 Defining Limits and Using Limit Notation

Name: ______ Class: ______ Date: ______ **Total: 9 marks**

KEY WORDS limit notation 极限记号 • arbitrarily 任意地 • limit 极限 • approaches 趋近

Objective

Build the skills to answer exam questions on **defining limits and using limit notation**.

You must be able to:

- read and write **limit notation 极限记号** $\lim_{x\to a} f(x) = L$
- explain that a limit need not equal $f(a)$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Limit notation**

$$\lim_{x\to a} f(x) = L$$

means $f(x)$ gets arbitrarily close to L as x approaches a from **both sides** — regardless of the value (or existence) of $f(a)$.

■ **Example**

$f(x) = x + 1$: $\lim_{x\to 2} f(x) = 3$, which here also equals $f(2) = 3$.

2 Practice

2.1 State what $\lim_{x\to a} f(x) = L$ means. [1]

2.2 State whether the limit must equal $f(a)$. [1]

2.3 For $f(x) = 2x$, state $\lim_{x\to 3} f(x)$. [2]

3 Exam-style questions

3.1 $\lim_{x\to a} f(x) = L$ means $f(x)$ approaches [1]

- **A** a
- **B** L as x approaches a
- **C** infinity
- **D** $f(a)$ always

3.2 The value of a limit [1]

- **A** always equals $f(a)$
- **B** may differ from $f(a)$
- **C** is always 0
- **D** never exists

3.3 $f(x) = x + 1$.

(a) State $\lim_{x\to 2} f(x)$. [1]

(b) State $f(2)$. [1]

(c) State whether they agree. [1]

Solutions

2.1 $f(x)$ gets arbitrarily close to L as x approaches a.

2.2 no — the limit may differ from $f(a)$.

2.3 $\lim_{x\to 3} 2x = 6$.

3.1 **B**.

3.2 **B**.

3.3 (a) 3. (b) 3. (c) yes.

AP CALCULUS AB • EXERCISE SHEET

1.3 Estimating Limit Values from Graphs

Name: ________ Class: ______ Date: ______ **Total: 9 marks**

KEY WORDS one-sided limits 单侧极限 • left-hand limit 左极限 • right-hand limit 右极限 • hole 空心点 • limit 极限

Objective

Build the skills to answer exam questions on **estimating limit values from graphs**.

You must be able to:

- read **one-sided limits** 单侧极限 from a graph
- decide whether the two-sided limit exists

1 Worked examples

Study these first. Each one shows the method for a question type used later.

> **Limits from graphs**
>
> Read the y-value the graph approaches from the left ($\lim_{x\to a^-}$) and from the right ($\lim_{x\to a^+}$).
>
> The two-sided limit $\lim_{x\to a} f(x)$ exists only if **both** one-sided limits equal the same L — even if there is a **hole** 空心点 at a.

2 Practice

2.1 State when a two-sided limit exists. [1]

2.2 State the notation for a left-hand limit. [1]

2.3 If the limit from the left is 4 and from the right is 4, state the two-sided limit. [2]

3 Exam-style questions

3.1 A two-sided limit exists if and only if [1]

- **A** $f(a)$ is defined
- **B** the left and right limits are equal
- **C** the graph is a straight line
- **D** the limit is 0

3.2 A hole in the graph at $x = a$ means [1]

- **A** the limit cannot exist
- **B** the limit may still exist
- **C** f is linear
- **D** $a = 0$

3.3 A graph approaches 5 from the left and 3 from the right at $x = 2$.

(a) State the left-hand limit. [1]

(b) State the right-hand limit. [1]

(c) State whether the two-sided limit exists. [1]

Solutions

2.1 when the left-hand and right-hand limits are equal.

2.2 $\lim_{x \to a^-} f(x)$.

2.3 the two-sided limit is 4.

3.1 **B**.

3.2 **B**.

3.3 (a) 5. (b) 3. (c) no, the one-sided limits differ.

AP CALCULUS AB • EXERCISE SHEET

1.4 Estimating Limit Values from Tables

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS table 表格 • limit 极限 • right-hand limit 右极限 • approaches 趋近

Objective

Build the skills to answer exam questions on **estimating limit values from tables**.

You must be able to:

- use a **table** 表格 of values approaching a to estimate a limit
- check the trend from both sides

1 Worked examples

Study these first. Each one shows the method for a question type used later.

> ■ **Limits from tables**
>
> Tabulate $f(x)$ for x close to a on **both sides**. If the values settle toward one number, that is the estimated limit.
>
> ■ **Example**
>
> $f(1.9) = 3.8$, $f(1.99) = 3.98$, $f(2.01) = 4.02$: the values approach 4, so $\lim_{x\to 2} f(x) \approx 4$.

2 Practice

2.1 State how a table is used to estimate a limit. [1]

2.2 From $f(1.9) = 3.8$, $f(1.99) = 3.98$, $f(2.01) = 4.02$, estimate $\lim_{x\to 2} f(x)$. [2]

2.3 State why you check both sides of a. [1]

3 Exam-style questions

3.1 To estimate a limit from a table, pick x-values [1]

- **A** far from a
- **B** approaching a from both sides
- **C** only greater than a
- **D** equal to a

3.2 If $f(x)$ approaches 7 from both sides, the limit is [1]

- **A** 0
- **B** 7
- **C** undefined
- **D** 14

3.3 Table: $f(2.9) = 5.9$, $f(2.99) = 5.99$, $f(3.01) = 6.01$.

(a) State the left-hand trend. [1]

(b) State the right-hand trend. [1]

(c) State $\lim_{x\to 3} f(x)$. [1]

Solutions

2.1 tabulate $f(x)$ for x near a and see what value the outputs approach.

2.2 the values approach 4, so $\lim_{x\to 2} f(x) \approx 4$.

2.3 to confirm the left- and right-hand limits agree.

3.1 **B**.

3.2 **B**.

3.3 (a) approaching 6. (b) approaching 6. (c) 6.

AP CALCULUS AB • EXERCISE SHEET

1.5 Determining Limits Using Algebraic Properties of Limits

Name: ________ Class: ______ Date: _______ **Total: 9 marks**

KEY WORDS continuous 连续 • direct substitution 直接代入 • limit laws 极限律 • limit 极限

Objective

Build the skills to answer exam questions on **determining limits using algebraic properties of limits**.

You must be able to:

- apply the **limit laws** 极限律 (sum, product, constant multiple)
- use **direct substitution** 直接代入 for continuous functions

1 Worked examples

Study these first. Each one shows the method for a question type used later.

> **Limit laws**
>
> Limits distribute over sums, differences, products, quotients (denominator $\neq 0$), and constant multiples. For a continuous function, $\lim_{x\to a} f(x) = f(a)$ (**direct substitution**).
>
> **Example**
>
> $\lim_{x\to 3}(x^2 - 1) = 3^2 - 1 = 8$.

2 Practice

2.1 State the limit of a sum $\lim(f + g)$. [1]

2.2 Evaluate $\lim_{x\to 2}(3x + 1)$ by substitution. [2]

2.3 State when direct substitution works. [1]

3 Exam-style questions

3.1 $\lim(f+g)$ equals [1]

- **A** $\lim f \cdot \lim g$
- **B** $\lim f + \lim g$
- **C** $\lim f - \lim g$
- **D** $f(a)$

3.2 For a continuous function, $\lim_{x\to a} f(x)$ equals [1]

- **A** 0
- **B** $f(a)$
- **C** ∞
- **D** a

3.3 Evaluate $\lim_{x\to 3}(x^2-1)$.

(a) Substitute $x=3$. [1]

(b) Compute 3^2. [1]

(c) State the limit. [1]

Solutions

2.1 $\lim f + \lim g$.

2.2 $3(2)+1=7$.

2.3 when the function is continuous at a.

3.1 **B**.

3.2 **B**.

3.3 (a) 3^2-1. (b) 9. (c) 8.

6. The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	0	2	3	6
$f(x)$	−1	3	8	5
$f'(x)$	−5	4	9	−2

Part A

Find $\lim\limits_{x\to 2}\dfrac{f(x)}{x}$, or state that the limit does not exist.

Part B

Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

Part C

Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

Part D

Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\,dt$.

i. Find $k'(x)$.

ii. Find $k''(3)$. Show the work that leads to your answer.

STOP

END OF EXAM

AP CALCULUS AB • EXERCISE SHEET

1.6 Determining Limits Using Algebraic Manipulation

Name: ____________ Class: __________ Date: __________ **Total: 9 marks**

KEY WORDS factor and cancel 因式分解并约分 • indeterminate form 未定式 • limit 极限 • direct substitution 直接代入

Objective

Build the skills to answer exam questions on **determining limits using algebraic manipulation**.

You must be able to:

- recognise an **indeterminate form** 未定式 $\frac{0}{0}$
- factor and cancel to evaluate the limit

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Algebraic manipulation

When direct substitution gives $\frac{0}{0}$, factor and cancel the common factor, then substitute.

■ Example

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

2 Practice

2.1 State what to do when substitution gives $\frac{0}{0}$. [1]

2.2 Simplify $\dfrac{x^2-9}{x-3}$. [1]

2.3 Evaluate $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$. [2]

3 Exam-style questions

3.1 If direct substitution gives $\frac{0}{0}$, you should [1]

- **A** conclude the limit is 0
- **B** factor and cancel
- **C** stop — no limit exists
- **D** divide by x

3.2 $\dfrac{x^2-1}{x-1}$ simplifies to [1]

- **A** $x-1$
- **B** $x+1$
- **C** x^2+1
- **D** 1

3.3 $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$.

(a) Factor the numerator. [1]

(b) Cancel $(x-1)$. [1]

(c) Substitute $x=1$. [1]

Solutions

2.1 factor and cancel the common factor, then substitute.

2.2 $x+3$ (for $x \neq 3$).

2.3 $\lim_{x\to 3}(x+3) = 6$.

3.1 **B**.

3.2 **B**.

3.3 (a) $(x-1)(x+1)$. (b) $x+1$. (c) 2.

1.7 Selecting Procedures for Determining Limits

Name: ____________ Class: ________ Date: __________ **Total: 8 marks**

KEY WORDS vertical asymptote 垂直渐近线 • indeterminate form 未定式 • limit 极限 • direct substitution 直接代入 • one-sided limits 单侧极限

Objective

Build the skills to answer exam questions on **selecting procedures for determining limits**.

You must be able to:

- choose the right method (substitution, factoring, one-sided) for a limit
- interpret what $\frac{0}{0}$ and $\frac{\text{nonzero}}{0}$ signal

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Choosing a method

- try **direct substitution** first;
- $\frac{0}{0}$ (an **indeterminate form** 未定式) → factor and cancel, or rationalize;
- $\frac{\text{nonzero}}{0}$ → check for a vertical asymptote and use one-sided limits.

Example

$\lim_{x\to 2}\dfrac{x-2}{x^2-4}$: substitution gives $\frac{0}{0}$, so cancel to $\dfrac{1}{x+2}=\frac{1}{4}$.

2 Practice

2.1 State the first method to try. [1]

2.2 State what a $\frac{0}{0}$ result signals. [1]

2.3 State what a $\frac{\text{nonzero}}{0}$ result signals. [1]

3 Exam-style questions

3.1 The first thing to try for a limit is [1]

- **A** factoring
- **B** direct substitution
- **C** rationalizing
- **D** a table

3.2 A $\frac{0}{0}$ form means [1]

- **A** the limit is 0
- **B** more algebra is needed
- **C** the limit is ∞
- **D** no limit exists

3.3 $\lim_{x\to 2}\dfrac{x-2}{x^2-4}$.

(a) Substitute to see the form. [1]

(b) Factor the denominator. [1]

(c) Simplify and evaluate. [1]

Solutions

2.1 direct substitution.

2.2 that more algebra (factoring/rationalizing) is needed.

2.3 a possible vertical asymptote — check one-sided limits.

3.1 **B**.

3.2 **B**.

3.3 (a) $\frac{0}{0}$. (b) $(x-2)(x+2)$. (c) $\dfrac{1}{x+2} = \frac{1}{4}$.

1.8 Determining Limits Using the Squeeze Theorem

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS squeeze theorem 夹逼定理 • limit 极限

Objective

Build the skills to answer exam questions on **the Squeeze Theorem**.

You must be able to:

- state the **Squeeze Theorem 夹逼定理**
- apply it when a function is bounded between two others with equal limits

1 Worked examples

Study these first. Each one shows the method for a question type used later.

> ■ **The Squeeze Theorem**
>
> If $g(x) \le f(x) \le h(x)$ near a and
>
> $$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L,$$
>
> then $\lim_{x\to a} f(x) = L$.
>
> ■ **Example**
>
> Since $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$ and both bounds $\to 0$ at 0, the middle limit is 0.

2 Practice

2.1 State the Squeeze Theorem. [1]

2.2 State what must be true of the two bounding limits. [1]

2.3 If $-x^2 \le f(x) \le x^2$ and both bounds $\to 0$ at 0, state $\lim_{x\to 0} f(x)$. [2]

3 Exam-style questions

3.1 The Squeeze Theorem requires $g \le f \le h$ and [1]

- **A** $\lim g = \lim h$
- **B** $g = h$ everywhere
- **C** $f = 0$
- **D** $h = 0$

3.2 If both bounds approach 3, then $\lim f$ is [1]

- **A** 0
- **B** 3
- **C** 6
- **D** undefined

3.3 $-x^2 \le f(x) \le x^2$ near 0.

(a) State $\lim_{x\to 0}(-x^2)$. [1]

(b) State $\lim_{x\to 0} x^2$. [1]

(c) State $\lim_{x\to 0} f(x)$. [1]

Solutions

2.1 if $g \le f \le h$ near a and g, h have the same limit L at a, then f also has limit L.

2.2 they must be equal (both equal L).

2.3 $\lim_{x\to 0} f(x) = 0$.

3.1 A.

3.2 B.

3.3 (a) 0. (b) 0. (c) 0.

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{\text{[illegible]}}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

   (a) Find $h'(2)$.

   (b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

   (c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim\limits_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim\limits_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

   (d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

STOP

END OF EXAM

AP CALCULUS AB • EXERCISE SHEET

1.9 Connecting Multiple Representations of Limits

Name: ____________ Class: __________ Date: __________ **Total: 8 marks**

KEY WORDS algebra 代数 • graph 图像 • table 表格 • limit 极限 • approaches 趋近

Objective

Build the skills to answer exam questions on **connecting multiple representations of limits**.

You must be able to:

- find a limit from a graph, a table, or algebra
- cross-check that the three representations agree

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Multiple representations

A limit can be found from a **graph** 图像, a **table** 表格, or **algebra** 代数 — and all three should give the same value. Cross-check when unsure.

Example

If algebra gives lim = 5, the table trend is → 5, and the graph approaches 5, the limit is 5.

2 Practice

2.1 Name three ways to find a limit. [1]

2.2 State what should be true across the representations. [1]

2.3 If algebra gives 5 and the table trend is 5, state the limit. [1]

3 Exam-style questions

3.1 A limit can be represented by [1]

- **A** only a graph
- **B** a graph, table, or algebra
- **C** only algebra
- **D** none of these

3.2 The three representations of a limit should [1]

- **A** conflict
- **B** all agree
- **C** give different values
- **D** be ignored

3.3 For a function, algebra gives $\lim = 4$, the table trend is $\to 4$, and the graph approaches 4.

(a) State the value from algebra. [1]

(b) State whether they agree. [1]

(c) State the limit. [1]

Solutions

2.1 from a graph, a table, or algebra.

2.2 they should all give the same value.

2.3 5.

3.1 **B**.

3.2 **B**.

3.3 (a) 4. (b) yes. (c) 4.

AP CALCULUS AB • EXERCISE SHEET

1.10 Exploring Types of Discontinuities

Name: __________ Class: ________ Date: _________ **Total: 9 marks**

KEY WORDS discontinuity 不连续点 • infinite discontinuity 无穷间断 • jump discontinuity 跳跃间断 • removable discontinuity 可去间断 • jump 跳跃

Objective

Build the skills to answer exam questions on **types of discontinuities**.

You must be able to:

- classify a **discontinuity** 不连续点 as removable, jump, or infinite

1 Worked examples

Study these first. Each one shows the method for a question type used later.

> ■ **Three types of discontinuity**
> - **removable** (a **hole**): the limit exists but $\neq f(a)$, or $f(a)$ is undefined;
> - **jump**: the left and right limits differ;
> - **infinite**: a vertical asymptote (the function blows up).

2 Practice

2.1 Name the three types of discontinuity. [1]

2.2 State which type is a "hole". [1]

2.3 State which type occurs at a vertical asymptote. [2]

3 Exam-style questions

3.1 A discontinuity where the left and right limits differ is a [1]

- **A** removable discontinuity
- **B** jump discontinuity
- **C** infinite discontinuity
- **D** point of continuity

3.2 A hole in a graph is a [1]

- **A** jump discontinuity
- **B** removable discontinuity
- **C** infinite discontinuity
- **D** point of continuity

3.3 Classify each.

(a) A vertical asymptote. [1]

(b) A single missing point. [1]

(c) A step where the graph jumps. [1]

Solutions

2.1 removable, jump, and infinite.

2.2 removable.

2.3 infinite.

3.1 **B**.

3.2 **B**.

3.3 (a) infinite. (b) removable. (c) jump.

1.11 Defining Continuity at a Point

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS continuity 连续性 • continuous 连续 • limit 极限 • discontinuous 间断

Objective

Build the skills to answer exam questions on **defining continuity at a point**.

You must be able to:

- state the three conditions for **continuity 连续性** at a point
- test continuity using $\lim_{x\to a} f(x) = f(a)$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Continuity at a point

f is continuous at a when all three hold:

1. $f(a)$ is defined;
2. $\lim_{x\to a} f(x)$ exists;
3. they are equal: $\lim_{x\to a} f(x) = f(a)$.

Example

If $\lim_{x\to 2} f(x) = 5$ and $f(2) = 5$, then f is continuous at 2.

2 Practice

2.1 State the three conditions for continuity at a. [1]

2.2 State the single equation summarizing continuity at a. [1]

2.3 If $\lim_{x \to 2} f(x) = 5$ and $f(2) = 5$, is f continuous at 2? [2]

3 Exam-style questions

3.1 f is continuous at a if and only if [1]

- **A** $f(a)$ is defined
- **B** $\lim_{x \to a} f(x) = f(a)$
- **C** the limit exists only
- **D** f is linear

3.2 If $\lim_{x \to a} f(x)$ exists but $\neq f(a)$, then f is [1]

- **A** continuous at a
- **B** discontinuous at a
- **C** linear
- **D** undefined everywhere

3.3 $f(2) = 5$ and $\lim_{x \to 2} f(x) = 5$.

(a) Is $f(2)$ defined? [1]

(b) Does the limit exist? [1]

(c) Is f continuous at 2? [1]

Solutions

2.1 $f(a)$ is defined; the limit exists; the limit equals $f(a)$.

2.2 $\lim_{x \to a} f(x) = f(a)$.

2.3 yes — the limit exists, $f(2)$ is defined, and they are equal.

3.1 B.

3.2 B.

3.3 (a) yes. (b) yes. (c) yes.

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{[illegible]}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \le h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

STOP

END OF EXAM

AP CALCULUS AB • EXERCISE SHEET

1.12 Confirming Continuity over an Interval

Name: ____________ Class: __________ Date: __________ **Total: 9 marks**

KEY WORDS interval 区间 • continuous on an interval 在区间上连续 • exponential 指数 • continuous 连续 • rational function 有理函数

Objective

Build the skills to answer exam questions on **confirming continuity over an interval**.

You must be able to:

- state what it means for a function to be continuous on an **interval** 区间
- know where standard functions are continuous

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Continuity over an interval

f is continuous on an interval if it is continuous at **every** point in it.

- polynomials, exponentials, sin, cos are continuous **everywhere**;
- a rational function is continuous except where its denominator is 0.

Example

$f(x) = \dfrac{1}{x - 3}$ is continuous everywhere except at $x = 3$.

2 Practice

2.1 State what continuity over an interval means. [1]

2.2 State where a polynomial is continuous. [1]

2.3 State where a rational function may be discontinuous. [2]

3 Exam-style questions

3.1 A function is continuous on an interval if it is continuous [1]

- **A** at one point
- **B** at every point of the interval
- **C** at the endpoints only
- **D** nowhere

3.2 Polynomials are continuous [1]

- **A** nowhere
- **B** everywhere
- **C** only at 0
- **D** only for $x > 0$

3.3 $f(x) = \dfrac{1}{x-3}$.

(a) Where is it discontinuous? [1]

(b) Is it continuous on $[0, 2]$? [1]

(c) Is it continuous on $[0, 4]$? [1]

Solutions

2.1 it is continuous at every point of the interval.

2.2 everywhere (for all real x).

2.3 where its denominator equals 0.

3.1 **B**.

3.2 **B**.

3.3 (a) $x = 3$. (b) yes (3 is not in $[0, 2]$). (c) no (3 is in $[0, 4]$).

AP CALCULUS AB • EXERCISE SHEET

1.13 Removing Discontinuities

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS removable discontinuity 可去间断点 • hole 空洞 • limit 极限 • continuous 连续 • jump 跳跃

Objective

Build the skills to answer exam questions on **removing discontinuities**.

You must be able to:

- remove a **removable discontinuity** 可去间断点 by redefining f at the hole

1 Worked examples

Study these first. Each one shows the method for a question type used later.

> ■ **Removing a discontinuity**
>
> A removable discontinuity (a hole) can be removed by defining $f(a) = \lim_{x\to a} f(x)$.
>
> ■ **Example**
>
> $f(x) = \dfrac{x^2-4}{x-2}$ has a hole at $x = 2$; since the limit there is 4, setting $f(2) = 4$ makes f continuous.

2 Practice

2.1 State how to remove a removable discontinuity. [1]

2.2 For $f(x) = \dfrac{x^2-4}{x-2}$, find the value that removes the hole at 2. [2]

2.3 State which kind of discontinuity can be removed. [1]

3 Exam-style questions

3.1 A removable discontinuity is removed by setting $f(a)$ equal to [1]

- **A** 0
- **B** the limit at a
- **C** $f(a+1)$
- **D** ∞

3.2 Which discontinuity can be removed? [1]

- **A** jump
- **B** infinite
- **C** removable
- **D** none

3.3 $f(x) = \dfrac{x^2-9}{x-3}$, with a hole at 3.

(a) Simplify f. [1]

(b) Find the limit at 3. [1]

(c) State $f(3)$ to remove the hole. [1]

Solutions

2.1 define $f(a)$ to equal the limit $\lim_{x\to a} f(x)$.

2.2 the limit is 4, so set $f(2) = 4$.

2.3 a removable discontinuity (a hole).

3.1 **B**.

3.2 **C**.

3.3 (a) $x + 3$ (for $x \neq 3$). (b) 6. (c) $f(3) = 6$.

1.14 Connecting Infinite Limits and Vertical Asymptotes

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS vertical asymptote 垂直渐近线 • infinite limit 无穷极限 • limit 极限 • horizontal asymptote 水平渐近线 • rational function 有理函数

Objective

Build the skills to answer exam questions on **connecting infinite limits and vertical asymptotes**.

You must be able to:

- relate an **infinite limit** 无穷极限 at a to a **vertical asymptote** 垂直渐近线 $x = a$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Infinite limits

If $f(x) \to \pm\infty$ as $x \to a$, the line $x = a$ is a **vertical asymptote**. This happens where the denominator $\to 0$ but the numerator does not.

Example

$f(x) = \dfrac{1}{x-2}$ has a vertical asymptote at $x = 2$; as $x \to 2^+$, $f(x) \to +\infty$.

2 Practice

2.1 State what an infinite limit at a indicates. [1]

2.2 For $f(x) = \dfrac{1}{x}$, state the vertical asymptote. [1]

2.3 State where a rational function has a vertical asymptote. [2]

3 Exam-style questions

3.1 If $\lim_{x\to a} f(x) = \infty$, then $x = a$ is a [1]

- **A** horizontal asymptote
- **B** vertical asymptote
- **C** hole
- **D** root

3.2 $f(x) = \dfrac{1}{x-2}$ has a vertical asymptote at [1]

- **A** $x = 0$
- **B** $x = 2$
- **C** $y = 2$
- **D** $x = -2$

3.3 $f(x) = \dfrac{1}{x-5}$.

(a) Where is the denominator 0? [1]

(b) State the vertical asymptote. [1]

(c) State the behaviour of $\lim_{x\to 5^+} f(x)$. [1]

Solutions

2.1 the graph has a vertical asymptote there.

2.2 $x = 0$.

2.3 where the denominator is 0 but the numerator is not.

3.1 **B**.

3.2 **B**.

3.3 (a) $x = 5$. (b) $x = 5$. (c) $+\infty$.

AP CALCULUS AB • EXERCISE SHEET

1.15 Connecting Limits at Infinity and Horizontal Asymptotes

Name: ______ Class: ______ Date: ______ **Total: 9 marks**

KEY WORDS horizontal asymptote 水平渐近线 • degrees 次数 • ratio of the leading coefficients 首项系数之比 • limit at infinity 无穷远处极限 • limit 极限

Objective

Build the skills to answer exam questions on **connecting limits at infinity and horizontal asymptotes**.

You must be able to:

- relate a **limit at infinity** 无穷远处极限 to a **horizontal asymptote** 水平渐近线
- find it from the ratio of leading terms

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Limits at infinity

$\lim_{x\to\infty} f(x) = L$ means $y = L$ is a horizontal asymptote. For a rational function, compare leading terms:

- degree(top) < degree(bottom) $\to y = 0$;
- degrees equal $\to y = \dfrac{\text{leading coeff of top}}{\text{leading coeff of bottom}}$;
- degree(top) > degree(bottom) $\to$ no horizontal asymptote.

Example

$\lim_{x\to\infty} \dfrac{1}{x} = 0$, so $y = 0$ is a horizontal asymptote.

2 Practice

2.1 State what $\lim_{x\to\infty} f(x) = L$ indicates. [1]

2.2 For $f(x) = \dfrac{1}{x}$, state $\lim_{x\to\infty} f(x)$. [1]

2.3 For $f(x) = \dfrac{2x}{x+1}$, state the horizontal asymptote. [2]

3 Exam-style questions

3.1 $\lim_{x\to\infty} f(x) = L$ means $y = L$ is a [1]

- **A** vertical asymptote
- **B** horizontal asymptote
- **C** hole
- **D** root

3.2 $\lim_{x\to\infty} \dfrac{1}{x}$ equals [1]

- **A** 1
- **B** 0
- **C** ∞
- **D** -1

3.3 $f(x) = \dfrac{3x^2}{x^2+1}$.

(a) Compare the degrees. [1]

(b) Take the ratio of the leading coefficients. [1]

(c) State the horizontal asymptote. [1]

Solutions

2.1 the graph has a horizontal asymptote $y = L$.

2.2 0.

2.3 $y = 2$ (equal degrees, ratio 2/1).

3.1 **B**.

3.2 **B**.

3.3 (a) equal (both degree 2). (b) $3/1 = 3$. (c) $y = 3$.

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25+t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

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6. Particle P moves along the x-axis such that, for time $t > 0$, its position is given by $x_P(t) = 6 - 4e^{-t}$.

Particle Q moves along the y-axis such that, for time $t > 0$, its velocity is given by $v_Q(t) = \frac{1}{t^2}$. At time $t = 1$, the position of particle Q is $y_Q(1) = 2$.

(a) Find $v_P(t)$, the velocity of particle P at time t.

(b) Find $a_Q(t)$, the acceleration of particle Q at time t. Find all times t, for $t > 0$, when the speed of particle Q is decreasing. Justify your answer.

(c) Find $y_Q(t)$, the position of particle Q at time t.

(d) As $t \to \infty$, which particle will eventually be farther from the origin? Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

AP CALCULUS AB • EXERCISE SHEET

1.16 Working with the Intermediate Value Theorem (IVT)

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS intermediate value theorem 介值定理 • change 变化 • continuous 连续

Objective

Build the skills to answer exam questions on **the Intermediate Value Theorem (IVT)**.

You must be able to:

- state the **Intermediate Value Theorem** 介值定理
- use it to show a value (e.g. a zero) is attained

1 Worked examples

Study these first. Each one shows the method for a question type used later.

The IVT

If f is **continuous** on $[a, b]$ and N is any value between $f(a)$ and $f(b)$, then there is at least one c in (a, b) with $f(c) = N$.

A sign change ($f(a)$ and $f(b)$ opposite signs) guarantees a **zero** in between.

Example

If f is continuous with $f(1) = -2$ and $f(3) = 4$, the IVT guarantees a zero in $(1, 3)$.

2 Practice

2.1 State the IVT. [1]

2.2 State the condition f must satisfy for the IVT. [1]

2.3 If $f(1) = -2$, $f(3) = 4$, and f is continuous, must f have a zero in $(1, 3)$? [2]

3 Exam-style questions

3.1 The IVT requires f to be [1]

- **A** differentiable
- **B** continuous on $[a, b]$
- **C** linear
- **D** a polynomial

3.2 If $f(2) = -1$ and $f(5) = 3$ (continuous), the IVT guarantees a c with [1]

- **A** $f(c) = 10$
- **B** $f(c) = 0$ for some c in $(2, 5)$
- **C** $f(c) = \infty$
- **D** no such c

3.3 f is continuous, $f(0) = -5$, $f(4) = 3$.

(a) State $f(0)$ and $f(4)$. [1]

(b) Note the sign change. [1]

(c) State what the IVT guarantees. [1]

Solutions

2.1 if f is continuous on $[a, b]$ and N lies between $f(a)$ and $f(b)$, then $f(c) = N$ for some c in (a, b).

2.2 it must be continuous on $[a, b]$.

2.3 yes — f is continuous and changes sign, so by the IVT there is a zero in $(1, 3)$.

3.1 **B**.

3.2 **B**.

3.3 (a) $f(0) = -5$, $f(4) = 3$. (b) it changes from negative to positive. (c) a zero (a c with $f(c) = 0$) in $(0, 4)$.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0,10]$, $[10,20]$, and $[20,30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t+15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS AB
SECTION II, Part A
Time—30 minutes

Number of problems—2

A graphing calculator is required for these problems.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	–120	–150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.