

8.11 Volumes by the Washer Method

Name: _____ Class: _____ Date: _____

Total: 14 marks

KEY WORDS washer 垫圈 • washer method 垫圈法 • disc 圆盘

Objective

Build the skills to answer exam questions on the **washer method** — volumes of revolution with a hole.

You must be able to:

- identify the **outer** and **inner** radii of a washer (ring)
- integrate $\pi(R_{\text{outer}}^2 - R_{\text{inner}}^2)$ (the **washer method** 垫圈法)
- set each radius as a distance from a curve to the axis

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ When there is a hole

If the region does **not** touch the axis, revolving it leaves a hole, so each slice is a **washer** (a ring). Its area is (outer disc) – (inner disc):

$$V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dx.$$

■ Identifying the radii

Revolve the region between $y = \sqrt{x}$ (top) and $y = x$ (bottom) on $[0, 1]$ about the x -axis. Outer radius (further curve) $R = \sqrt{x}$; inner radius $r = x$:

$$V = \pi \int_0^1 ((\sqrt{x})^2 - x^2) dx = \pi \int_0^1 (x - x^2) dx = \pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}.$$

■ Do not square then subtract wrongly

$R^2 - r^2$ is **not** $(R - r)^2$. Square each radius separately, then subtract.

■ Choosing outer vs inner

The **outer** radius comes from the curve **farther** from the axis; the **inner** from the curve **nearer** the axis. A quick sketch settles which is which.

2 Practice

Now apply the methods above.

2.1 State the washer-method formula for the volume. [1]

2.2 For $R = 4$ and $r = 2$, find $\pi(R^2 - r^2)$. [1]

2.3 Evaluate $\pi \int_0^1 (x - x^2) dx$. [2]

3 Exam-style questions

3.1 The washer method integrates [1]

- A $\pi \int (R - r)^2 dx$
- B $\pi \int (R^2 - r^2) dx$
- C $\pi \int R^2 dx$
- D $\int (R^2 - r^2) dx$

3.2 The region between $y = x$ and $y = x^2$ (from $x = 0$ to $x = 1$) is revolved about the x -axis.

(a) Identify the outer and inner radii. [2]

(b) Set up and evaluate the volume. [4]

3.3 The region bounded by $y = 4$ and $y = x^2$ (for $0 \leq x \leq 2$) is revolved about the x -axis. Set up the washer integral for the volume. [3]

Solutions

2.1 $V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dx.$

2.2 $\pi(16 - 4) = 12\pi.$

2.3 $\pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}.$

3.1 B — subtract the **squares** of the radii.

3.2 (a) On $(0, 1)$, $y = x$ is above $y = x^2$; outer $R = x$, inner $r = x^2$. (b) $V = \pi \int_0^1 (x^2 - x^4) dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$

3.3 Outer $R = 4$, inner $r = x^2$; $V = \pi \int_0^2 (4^2 - (x^2)^2) dx = \pi \int_0^2 (16 - x^4) dx.$