AP CALCULUS AB • EXERCISE SHEET

6.9 Integrating Using Substitution

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS definite integral 定积分

Objective

Build the skills to answer exam questions on **integration by substitution** (u-substitution) — reversing the chain rule.

You must be able to:

- choose $u = g(x)$ so that its derivative $g'(x)$ also appears in the integrand
- convert dx to du and integrate in terms of u
- handle a **definite** integral by changing the limits or converting back to x

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Choosing u

Look for an "inside function" whose derivative is also present. For $\int 2x\,(x^2+1)^3\,dx$, let $u = x^2+1$, so $du = 2x\,dx$. The integral becomes

$$\int u^3\,du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{4} + C.$$

Adjusting for a constant

If the derivative is off by a constant, fix it. For $\int x\,(x^2+1)^3\,dx$, let $u = x^2+1$, $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$:

$$\int u^3 \cdot \tfrac{1}{2}\,du = \frac{u^4}{8} + C.$$

Definite integral — change the limits

For $\int_0^2 2x(x^2+1)^3\,dx$, with $u = x^2+1$: when $x = 0$, $u = 1$; when $x = 2$, $u = 5$. So

$$\int_1^5 u^3\,du = \left[\tfrac{u^4}{4}\right]_1^5 = \frac{625-1}{4} = 156.$$

A trig example

$\int \cos(3x)\,dx$: let $u = 3x$, $du = 3\,dx$, so $dx = \frac{1}{3}\,du$, giving $\frac{1}{3}\sin(3x) + C$.

2 Practice

Now apply the methods above.

2.1 For $\int 3x^2(x^3+5)^4\,dx$, state a good choice of u and find du. [2]

2.2 Find $\int 2x\,e^{x^2}\,dx$. [2]

2.3 Find $\int \sin(4x)\,dx$. [2]

3 Exam-style questions

3.1 To integrate $\int x\cos(x^2)\,dx$, the best substitution is [1]

- **A** $u = x$
- **B** $u = \cos x$
- **C** $u = x^2$
- **D** $u = \cos(x^2)$

3.2 Find $\int (2x+1)(x^2+x)^5\,dx$. [3]

3.3 Evaluate $\int_0^1 3x^2\sqrt{x^3+1}\,dx$ using a substitution, changing the limits. [4]

Solutions

2.1 $u = x^3 + 5$; $du = 3x^2\,dx$.

2.2 $u = x^2$, $du = 2x\,dx$; $\int e^u\,du = e^{x^2} + C$.

2.3 $u = 4x$, $dx = \frac{1}{4}du$; $-\frac{1}{4}\cos(4x) + C$.

3.1 C — $u = x^2$ gives $du = 2x\,dx$, matching the x in the integrand.

3.2 $u = x^2 + x$, $du = (2x+1)\,dx$; $\int u^5\,du = \frac{u^6}{6} = \dfrac{(x^2+x)^6}{6} + C$.

3.3 $u = x^3 + 1$, $du = 3x^2\,dx$; limits $x = 0 \to u = 1$, $x = 1 \to u = 2$; $\int_1^2 \sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_1^2$ $= \frac{2}{3}(2\sqrt{2} - 1) \approx 1.22$.