AP CALCULUS AB • EXERCISE SHEET

6.10 Long Division and Completing the Square

Name: _____ Class: _____ Date: _____ **Total: 14 marks**

Objective

Build the skills to answer exam questions on **integrating with long division and completing the square** —rewriting a rational integrand into an integrable form.

You must be able to:

- use **polynomial long division** 多项式长除法 when the numerator's degree is $\geq$ the denominator's
- **complete the square** 配方 in a denominator to reach an **arctangent** form
- recognise $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ **Long division first**

$\displaystyle\int \frac{x^2}{x+1}\,dx$ is top-heavy. Divide: $\displaystyle\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. Then

$$\int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{x^2}{2} - x + \ln|x+1| + C.$$

■ **A simple 1/x-type**

$\displaystyle\int \frac{1}{x-3}\,dx = \ln|x-3| + C$ (substitution $u = x-3$). Watch for these after dividing.

■ **Completing the square**

$\displaystyle\int \frac{1}{x^2+4x+5}\,dx$: complete the square, $x^2+4x+5 = (x+2)^2+1$. With $u = x+2$,

$$\int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+2) + C.$$

■

$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$. For $\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\arctan\frac{x}{3} + C$.

2 Practice

Now apply the methods above.

2.1 Use long division to rewrite $\frac{x+3}{x+1}$ as $1 + \frac{?}{x+1}$. [2]

2.2 Find $\int \frac{1}{x^2+16}\,dx$. [2]

2.3 Complete the square: write $x^2+6x+13$ in the form $(x+a)^2+b$. [2]

3 Exam-style questions

3.1 $\int \frac{1}{x^2+25}\,dx =$ [1]

- **A** $\arctan(x) + C$
- **B** $\frac{1}{5}\arctan\frac{x}{5} + C$
- **C** $\ln(x^2+25) + C$
- **D** $5\arctan(5x) + C$

3.2 Find $\int \frac{x+2}{x+1}\,dx$. [3]

3.3 Find $\int \frac{1}{x^2+2x+10}\,dx$ by completing the square. [4]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **6.10 Long Division and Completing the Square** lesson on the **Learn** page;
- read the **Integrating Using Long Division and Completing the Square** section of the AP Calculus AB handout on the **Know** page.

Solutions

2.1 $\dfrac{x+3}{x+1} = 1 + \dfrac{2}{x+1}$.

2.2 $\frac{1}{4}\arctan\frac{x}{4} + C$.

2.3 $(x+3)^2 + 4$.

3.1 **B** —with $a = 5$, $\int \frac{dx}{x^2+25} = \frac{1}{5}\arctan\frac{x}{5} + C$.

3.2 $\dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$; $\int \left(1 + \frac{1}{x+1}\right)dx = x + \ln|x+1| + C$.

3.3 $x^2+2x+10 = (x+1)^2+9$; with $u = x+1$, $\int \frac{du}{u^2+9} = \frac{1}{3}\arctan\frac{u}{3} = \frac{1}{3}\arctan\dfrac{x+1}{3} + C$.