5.12 Behaviors of Implicit Relations

Name: ____________ Class: ________ Date: __________ **Total: 10 marks**

Objective

Build the skills to answer exam questions on the **behavior of implicit relations** — using $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ found by implicit differentiation to locate features of a curve.

You must be able to:

- find where an implicit curve has a **horizontal tangent** 水平切线 ($\frac{dy}{dx} = 0$) or a **vertical tangent** 垂直切线 ($\frac{dy}{dx}$ undefined)
- decide **concavity** 凹凸性 from the sign of $\dfrac{d^2y}{dx^2}$ on the curve
- evaluate these using a point that lies **on** the curve

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

Horizontal tangents

For $x^2 + y^2 = 25$, implicit differentiation gives $2x + 2y\dfrac{dy}{dx} = 0$, so

$$\frac{dy}{dx} = -\frac{x}{y}.$$

A **horizontal** tangent needs $\dfrac{dy}{dx} = 0$, i.e. the numerator $x = 0$ (with $y \neq 0$): the points $(0, 5)$ and $(0, -5)$.

Vertical tangents

A **vertical** tangent occurs where $\dfrac{dy}{dx}$ is undefined —the denominator zero, $y = 0$ (with $x \neq 0$): the points $(5, 0)$ and $(-5, 0)$. This matches the circle's shape.

Concavity on an implicit curve

Differentiate $\dfrac{dy}{dx} = -\dfrac{x}{y}$ again (quotient + chain rule):

$$\frac{d^2y}{dx^2} = -\frac{y - x\,\frac{dy}{dx}}{y^2} = -\frac{y - x(-x/y)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}.$$

> Where $y > 0$ this is negative (**concave down**, the top of the circle); where $y < 0$ it is positive (**concave up**).

2 Practice

Now apply the methods above.

2.1 For $x^2 + y^2 = 25$, state $\dfrac{dy}{dx}$. [1]

2.2 Find the point(s) on $x^2 + y^2 = 25$ where the tangent is **horizontal**. [2]

2.3 A curve satisfies $\dfrac{dy}{dx} = \dfrac{2x}{y}$. State where its tangent line is **vertical**. [1]

3 Exam-style questions

3.1 A curve has $\dfrac{dy}{dx} = \dfrac{x-1}{y}$. A horizontal tangent occurs where [1]

- **A** $y = 0$
- **B** $x = 0$
- **C** $x = 1$
- **D** $x = y$

3.2 The curve $x^2 + xy + y^2 = 7$ passes through $(1, 2)$. Its derivative is $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$.

(a) Find the slope of the tangent at $(1, 2)$. [2]

(b) Determine whether the tangent at $(1, 2)$ is horizontal, vertical, or neither. [1]

3.3 For $x^2 + y^2 = 25$ it can be shown that $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$. State, with a reason, whether the curve is concave up or concave down at the point $(3, 4)$. [2]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **5.12 Behaviors of Implicit Relations** lesson on the **Learn** page;
- read the **Exploring Behaviors of Implicit Relations** section of the AP Calculus AB handout on the **Know** page.

Solutions

2.1 $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

2.2 Set numerator $x = 0$; points $(0, 5)$ and $(0, -5)$.

2.3 Vertical where denominator $y = 0$, i.e. at $(\pm 5, 0)$ —where the curve meets the x-axis.

3.1 **C** —a horizontal tangent needs the numerator zero, $x - 1 = 0$, so $x = 1$.

3.2 (a) $\dfrac{dy}{dx} = -\dfrac{2(1) + 2}{1 + 2(2)} = -\dfrac{4}{5}$. (b) Neither —the slope $-\frac{4}{5}$ is finite and non-zero.

3.3 $\dfrac{d^2y}{dx^2} = -\dfrac{25}{4^3} = -\dfrac{25}{64} < 0$; negative, so the curve is **concave down** at $(3, 4)$.