AP CALCULUS AB • EXERCISE SHEET

5.11 Solving Optimization Problems

Name: ____________ Class: ________ Date: __________ **Total: 17 marks**

Objective

Build the skills to answer exam questions on **solving optimization problems** —carrying the setup through to a justified maximum or minimum.

You must be able to:

- differentiate the **objective function** 目标函数 and solve $f'(x) = 0$ for the **critical points** 临界点
- **justify** that a critical point is a maximum or minimum (sign of f', or the second-derivative test, or the closed-interval candidates)
- give the answer the question asks for, with units

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ The full optimization method

Maximize the pen area $A(y) = 40y - 2y^2$ on $0 < y < 20$.

1. Differentiate: $A'(y) = 40 - 4y$.
2. Solve $A'(y) = 0$: $y = 10$.
3. Justify: $A''(y) = -4 < 0$, so $y = 10$ gives a **maximum**.
4. Answer: $x = 40 - 2(10) = 20$, so $A = 20 \times 10 = 200$ m^2.

■ Justifying with the first-derivative sign

If the second derivative is awkward, test the sign of f' on each side of the critical point. Here $A'(9) = +4 > 0$ and $A'(11) = -4 < 0$: f rises then falls, so the critical point is a **maximum**. A justification is **required** for full marks.

■ Minimizing a cost

A can's surface area is $S(r) = 2\pi r^2 + \dfrac{710}{r}$. Then $S'(r) = 4\pi r - \dfrac{710}{r^2}$. Setting $S' = 0$: $4\pi r^3 = 710$, so $r = \left(\dfrac{710}{4\pi}\right)^{1/3} \approx 3.84$ cm. Since $S''(r) = 4\pi + \dfrac{1420}{r^3} > 0$, this is a **minimum**.

2 Practice

Now apply the methods above.

2.1 For $A(y) = 40y - 2y^2$, find $A'(y)$ and solve $A'(y) = 0$. [2]

2.2 The area function is $A = 12x - x^2$. Find the width x that maximizes the area, and **justify** it is a maximum. [3]

2.3 A function $C(x) = x^2 + \frac{16}{x}$ models a cost for $x > 0$. Find the x that minimizes C. [3]

3 Exam-style questions

3.1 A critical point of f has $f'(c) = 0$ and $f''(c) < 0$. The point is a [1]

- **A** local minimum
- **B** local maximum
- **C** point of inflection
- **D** endpoint

3.2 A rectangular field of area $A = x(200 - x)$ m^2 is enclosed, where $0 < x < 200$.

(a) Find the value of x that maximizes A. [2]

(b) Justify that your value gives a maximum. [1]

(c) State the maximum area. [1]

3.3 An open box is made from card, with volume $V(x) = 150x - \frac{1}{4}x^3$ for $0 < x < \sqrt{600}$.

(a) Show that $V'(x) = 150 - \frac{3}{4}x^2$. [1]

(b) Find the value of x that maximizes the volume, and justify it. [3]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **5.11 Solving Optimization Problems** lesson on the **Learn** page;
- read the **Solving Optimization Problems** section of the AP Calculus AB handout on the **Know** page.

Solutions

2.1 $A'(y) = 40 - 4y$; $40 - 4y = 0 \Rightarrow y = 10$.

2.2 $A'(x) = 12 - 2x = 0 \Rightarrow x = 6$; $A''(x) = -2 < 0$, so $x = 6$ is a maximum.

2.3 $C'(x) = 2x - \dfrac{16}{x^2} = 0$; $2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x = 2$.

3.1 B —$f'(c) = 0$ with $f''(c) < 0$ is a local maximum.

3.2 (a) $A = 200x - x^2$, $A'(x) = 200 - 2x = 0 \Rightarrow x = 100$. (b) $A''(x) = -2 < 0$, so it is a maximum. (c) $A = 100(100) = 10\,000$ m^2.

3.3 (a) $V'(x) = 150 - \frac{3}{4}x^2$. (b) $150 - \frac{3}{4}x^2 = 0 \Rightarrow x^2 = 200 \Rightarrow x = \sqrt{200} \approx 14.1$; $V''(x) = -\frac{3}{2}x < 0$ for $x > 0$, so it is a maximum.