# Past-paper walk-through

## Volume with Washer Method: Revolving Around Other Axes ▪ 8.12

eXercise

# Questions in this set

- 2025 Q2
- 2024 Q6
- 2019 Q5

# Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

# Question 2

2025 ▪ Q2

The shaded region $R$ is bounded by the graphs of the functions $f$ and $g$, where $f(x) = x^2 - 2x$ and $g(x) = x + \sin(\pi x)$, as shown in the figure.
[Figure: In the $xy$-plane, the graph of $y = g(x)$ and the graph of $y = f(x)$ are shown. The shaded region $R$, labeled $R$, lies between the two curves. The point $(3, 3)$ is marked where the curves meet. The $y$-axis is labeled with values $-2, -1, 1, 2, 3, 4$ and the $x$-axis is labeled with values $1, 2, 3$, with origin $O$.]
(Note: Your calculator should be in radian mode.)

**(a)** Find the area of $R$. Show the setup for your calculations.

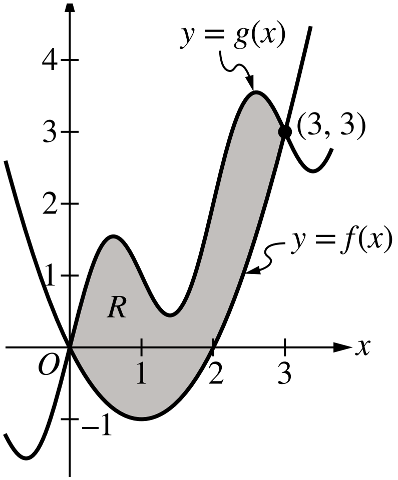
**(b)** Region $R$ is the base of a solid. For this solid, at each $x$ the cross section perpendicular to the $x$-axis is a rectangle with height $x$ and base in region $R$. Find the volume of the solid. Show the setup for your calculations.

# Question 2

**(c)** Write, but do not evaluate, an integral expression for the volume of the solid generated when the region $R$ is rotated about the horizontal line $y = -2$.

**(d)** It can be shown that $g'(x) = 1 + \pi\cos(\pi x)$. Find the value of $x$, for $0 < x < 1$, at which the line tangent to the graph of $f$ is parallel to the line tangent to the graph of $g$.

# Question 2



# Solution 2

## (a)

**Mark scheme**

- Area $= \int_0^3 (g(x) - f(x))\,dx = 5.136620 \approx 5.137$ (exact value $\dfrac{4+9\pi}{2\pi}$). Point 1: presents an integral of $g(x) - f(x)$, $|g(x) - f(x)|$, $f(x) - g(x)$, or $|f(x) - g(x)|$ over $[0, 3]$ (or an equivalent difference of definite integrals of $g$ and $f$). Point 2: correct answer 5.137 (or 5.136), with or without supporting work.

# Solution 2

## (b)

**Mark scheme**

- Volume $= \int_0^3 x(g(x) - f(x))\, dx = 7.704930 \approx 7.705$ (exact value $\dfrac{12+27\pi}{4\pi}$). Point 3: presents a definite integral with integrand a product of two nonconstant factors, one factor being $x$, $g(x) - f(x)$, or $f(x) - g(x)$. Point 4: correct answer 7.705 (or 7.704), with or without supporting work.

# Solution 2

## (c)

**Mark scheme**

- Volume $= \pi \int_0^3 \left[(g(x) - (-2))^2 - (f(x) - (-2))^2\right] dx =$ $\pi \int_0^3 \left[(g(x) + 2)^2 - (f(x) + 2)^2\right] dx$ (expression only, not evaluated). Point 5: integrand of the form $R^2 - r^2$ (or $|R^2 - r^2|$) where one of $R, r$ is $g$ or $f$ (or a difference involving a nonzero constant) and the other is correct or a difference between $f/g$ and a nonzero constant. Point 6: correct integrand $(g(x) + 2)^2 - (f(x) + 2)^2$ (or equivalent, e.g. reversed with a resolved sign). Point 7: correct limits of integration (0 to 3), the constant $\pi$, and the differential $dx$ all present and correctly combined; requires Point 5.

# Solution 2

**(d)**

## Mark scheme

- Set $f'(x) = g'(x)$: $2x - 2 = 1 + \pi\cos(\pi x) \Rightarrow x = 0.675819 \approx 0.676$ (or 0.675). Point 8: general statement $f'(x) = g'(x)$, or any correct equation substituting $2x - 2$ for $f'(x)$ and/or $1 + \pi\cos(\pi x)$ for $g'(x)$. Point 9: correct answer $x \approx 0.676$, with or without supporting work.

# Question 6

2024 ▪ Q6

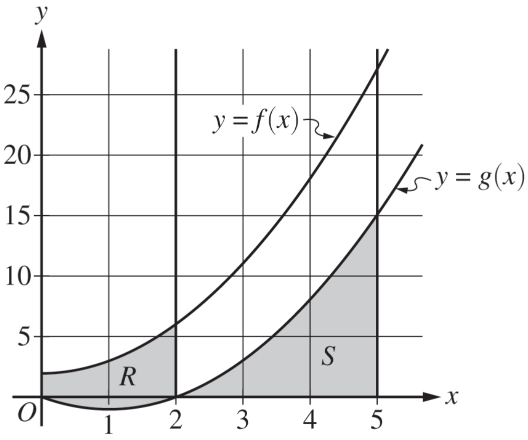
[Graph of $f$ and $g$; $x$-axis labeled 0 to 5, $y$-axis labeled 5 to 25; curve $y = f(x)$ is a upward-opening parabola-like curve passing through about $(0, 2)$ and rising steeply to about $(5, 27)$; curve $y = g(x)$ passes through the origin area, dips slightly below the $x$-axis between $x = 0$ and $x = 2$, then rises through about $(5, 15)$; the shaded region $R$ lies between the two curves from $x = 0$ to $x = 2$; the shaded region $S$ lies between the graph of $g$ and the $x$-axis from $x = 2$ to $x = 5$.]
The functions $f$ and $g$ are defined by $f(x) = x^2 + 2$ and $g(x) = x^2 - 2x$, as shown in the graph.
**(a)** Let $R$ be the region bounded by the graphs of $f$ and $g$, from $x = 0$ to $x = 2$, as shown in the graph. Write, but do not evaluate, an integral expression that gives the area of region $R$.

# Question 6

**(b)** Let $S$ be the region bounded by the graph of $g$ and the $x$-axis, from $x = 2$ to $x = 5$, as shown in the graph. Region $S$ is the base of a solid. For this solid, at each $x$ the cross section perpendicular to the $x$-axis is a rectangle with height equal to half its base in region $S$. Find the volume of the solid. Show the work that leads to your answer.

**(c)** Write, but do not evaluate, an integral expression that gives the volume of the solid generated when region $S$, as described in part (b), is rotated about the horizontal line $y = 20$.

# Question 6



# Solution 6

## (a)

**Mark scheme**

- With $f(x) = x^2 + 2$ and $g(x) = x^2 - 2x$, the area of region $R$ (bounded by $f$, $g$, from $x = 0$ to $x = 2$, where $f \geq g$) is given by the (unevaluated) integral Area $= \int_0^2 (f(x) - g(x))\, dx$. 1 point for an integrand equivalent to $f(x) - g(x)$ (or $|f(x) - g(x)|$, or a difference of two separate integrals with integrands $f$ and $g$), 1 point for the fully correct expression $\int_0^2 (f(x) - g(x))\, dx$ (equivalent forms accepted, e.g. $\int_0^2 f(x)\, dx - \int_0^2 g(x)\, dx$). Do not evaluate.

# Solution 6

## (b) Mark scheme

- Cross sections perpendicular to the $x$-axis over region $S$ ($g$ and the $x$-axis, $x = 2$ to $x = 5$) are rectangles with height $= \frac{1}{2}\cdot$base, so area $= \frac{1}{2}(g(x))^2$: $V = \int_2^5 \dfrac{1}{2}(g(x))^2\,dx = \int_2^5 \dfrac{1}{2}(x^2 - 2x)^2\,dx = \dfrac{1}{2}\int_2^5 (x^4 - 4x^3 + 4x^2)\,dx =$ $\dfrac{1}{2}\left[\dfrac{x^5}{5} - x^4 + \dfrac{4x^3}{3}\right]_2^5 = \dfrac{1}{2}\left(\dfrac{500}{3} - \dfrac{16}{15}\right) = \dfrac{414}{5} = 82.8$. 1 point for the integrand of the form $k(g(x))^2$, 1 point for the correct limits $x = 2$ to $x = 5$ with an integrand of the form $a(x)\cdot g(x)$, 1 point for a correct antiderivative, 1 point for the final numeric answer $414/5$ (or 82.8).

# Solution 6

## (c)

**Mark scheme**

- Rotating region $S$ about $y = 20$ gives washers with outer radius $20 - 0 = 20$ (from $y = 20$ down to the $x$-axis, $y = 0$) and inner radius $20 - g(x)$: $V = \pi \int_2^5 \left[20^2 - (20 - g(x))^2\right] dx = \pi \int_2^5 \left[400 - (20 - g(x))^2\right] dx = \pi \int_2^5 \left[400 - \left(20 - (x^2 - 2x)\right)^2\right] dx$ (do not evaluate). 1 point for the correct form of the integrand $20^2 - (20 - g(x))^2$ (or an equivalent difference-of-squares form), 1 point for substituting $g(x) = x^2 - 2x$ correctly into that integrand, 1 point for the correct limits $x = 2$ to $x = 5$ together with the constant $\pi$.

# Question 5

2019 ▪ Q5

Let $R$ be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the $y$-axis, and the vertical line $x = 2$, as shown in the figure below.

[Graph showing region $R$ shaded gray; $x$-axis from $O$ to past $2$ (gridlines at 1 and 2), $y$-axis with gridline at 1. The upper boundary curve $h(x)$ starts above the $y$-axis, rises to a peak between $x = 0$ and $x = 1$, then descends, meeting the vertical line $x = 2$ near $y \approx 1.5$-$2$. The lower boundary curve $g(x)$ starts near $(0, 1)$, dips down crossing the $x$-axis around $x = 1$, continues down to a minimum, then rises back up to meet $h(x)$ at the line $x = 2$. Region $R$ is the shaded area between the two curves from $x = 0$ to $x = 2$, bounded on the left by the $y$-axis and on the right by the line $x = 2$.]

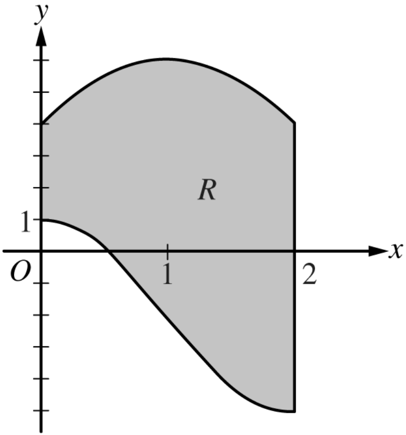
**(a)** Find the area of $R$.

# Question 5

**(b)** Region $R$ is the base of a solid. For the solid, at each $x$ the cross section perpendicular to the $x$-axis has area $A(x) = \dfrac{1}{x+3}$. Find the volume of the solid.

**(c)** Write, but do not evaluate, an integral expression that gives the volume of the solid generated when $R$ is rotated about the horizontal line $y = 6$.

# Question 5



# Solution 5

## (a) Mark scheme

- Area of $R = \int_0^2 (h(x) - g(x))\, dx = \int_0^2 \left[\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right] dx = \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2} = \left(\left(12 - \frac{2}{3}\right) - (-4+0)\right) - \left(\left(0 + \frac{2}{3}\right) - (0+0)\right) = 12 - \frac{2}{3} + 4 - \frac{2}{3} = \frac{44}{3}$. The area of $R$ is $\frac{44}{3}$. Points: 1 for the integrand (correctly set up as $h(x) - g(x)$), 1 for the antiderivative of $3\cos\left(\frac{\pi}{2}x\right)$, 1 for the antiderivative of the remaining terms, 1 for the final answer $\frac{44}{3}$.

# Solution 5

## (b)

**Mark scheme**

- Volume $= \int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx = \left[\ln(x+3)\right]_{x=0}^{x=2} = \ln 5 - \ln 3$. The volume of the solid is $\ln 5 - \ln 3$. Points: 1 for the correct integral, 1 for the final answer $\ln 5 - \ln 3$.

# Solution 5

## (c)

### Mark scheme

- Volume when $R$ is rotated about the line $y = 6$ (washer method), written but NOT evaluated: $\pi \int_0^2 \left[(6 - g(x))^2 - (6 - h(x))^2\right] dx$. Points: 1 for correct limits and constant ($\pi$, from $0$ to $2$), 1 for the correct form of the integrand (outer-radius-squared minus inner-radius-squared, using $6 - g(x)$ and $6 - h(x)$), 1 for the fully correct integrand overall.